MAS334 COMBINATORICS — PROBLEM SHEET 1

Please hand in Exercises 1.1 and 1.3 by the end of Week 2.

Exercise 1.1. Use the Binomial Theorem to show that, for any positive integer n,

(A)
$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = 2^n$$

(B)
$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots + (-1)^n \binom{n}{n} = 0$$

(C)
$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots = 2^{n-1}$$

(D)
$$\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots = 2^{n-1}$$

Exercise 1.2.

(a) If you draw n (infinite) straight lines in the plane with no two parallel and no three meeting at a point, how many intersection points will there be altogether?

(b) Now n straight lines are drawn in the plane consisting precisely of x_1 parallel in one direction, x_2 parallel in another direction, ... and x_k parallel in another direction, and with no three meeting at a point. By considering the number of intersection points lost by having parallel lines, show that the number of intersection points will be

$$\frac{1}{2}\,(n^2 - x_1^2 - x_2^2 - \ldots - x_k^2).$$

(c) Draw 14 straight lines in the plane, with no three meeting at a point, so that there are 61 intersection points altogether.

Exercise 1.3. By considering colouring k items out of n items using either red or blue to colour each item, show that

$$\sum_{j=0}^{k} \binom{n}{j}\binom{n-j}{k-j} = \binom{n}{0}\binom{n}{k} + \binom{n}{1}\binom{n-1}{k-1} + \binom{n}{2}\binom{n-2}{k-2} + \cdots + \binom{n}{k}\binom{n-k}{0} = 2^k\binom{n}{k}.$$

Exercise 1.4.

(i) Show by each of the following methods that

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

  (a) Consider the choice of n people from a group consisting of n men and n women.
  (b) Use the expansion of $(1+x)^{2n}$.
  (c) Count the number of routes in a suitable grid.

(ii) Given a collection of $2n$ people consisting of n men and n women, how many ways can a subset be chosen, the only restriction being that the number of women chosen equals the number of men chosen?

(iii) By considering the number of ways of choosing a subset of a set of n people and one person as a leader in the subset, or otherwise, show that

$$\binom{n}{1} + 2\binom{n}{2} + \cdots + n\binom{n}{n} = n\,2^{n-1}.$$

Exercise 1.5. Consider the equation $w_1 + \cdots + w_{10} = 57$, with $w_i \geq i$ for all i. How many solutions are there?

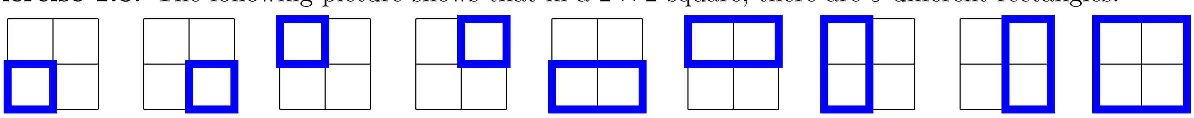
Exercise 1.6. The monster size Toblerone bar weighs 4.5kg, is 80cm long and contains 12 triangles.

(a) Suppose we want to share the bar among four people, by breaking it into four pieces in the obvious way, but the pieces are allowed to be of different sizes. How many ways are there to do this?

(b) Suppose instead that we want to divide the bar into pieces, but we do not specify in advance how many pieces there should be (and we allow the degenerate case where there is just one piece). How many ways are there to do this?

Exercise 1.7. Consider the equation $u_1 + \cdots + u_{10} = 5 \pmod{10}$, with $0 \leq u_i < 10$. How many solutions are there? (This is easier than Proposition 2.1.)

Exercise 1.8. The following picture shows that in a 2×2 square, there are 9 different rectangles.



How many rectangles are there in an $n \times n$ square?

(It is easiest to see this by thinking about the sides of the rectangles. If you have trouble spotting a solution, you should begin by considering small examples. Work out what the answer is in the $n \times n$ case, where $n = 1, 2, 3, 4, 5$, by directly counting. Make sure you do this carefully and accurately, as mistakes here will lead to you failing to spot a pattern. Now look for a pattern and *guess* the general answer based on this. Then look for an argument that justifies your guess.)

Exercise 1.9. Fix an integer $n > 0$. At some point you have probably seen the identity

$$\sum_{k=1}^{n} k^3 = \binom{n+1}{2}^2 = \frac{n^2(n+1)^2}{4}.$$

The usual proof (which is not hard) is by induction. Here we will instead give a bijective proof (due to Benjamin and Orrison).

(a) Put $S = \{(h,i,j,k) \mid 1 \leq h,i,j \leq k \leq n\}$. Show that $|S| = \sum_{k=1}^{n} k^3$.

(b) Put $T_0 = \{(a,b) \mid 1 \leq a \leq b \leq n\}$ and $T = \{(a,b,c,d) \mid 1 \leq a \leq b \leq n,\ 1 \leq c \leq d \leq n\}$. Show that $|T_0| = \binom{n+1}{2}$ and $|T| = \binom{n+1}{2}^2$.

(c) Define $f(h,i,j,k)$ (for $(h,i,j,k) \in S$) and $g(a,b,c,d)$ (for $(a,b,c,d) \in T$) as follows:

$$f(h,i,j,k) = \begin{cases} (h,i,j,k) & \text{if } h \leq i \\ (j,k,i,h-1) & \text{if } h > i \end{cases} \qquad g(a,b,c,d) = \begin{cases} (a,b,c,d) & \text{if } b \leq d \\ (d+1,c,a,b) & \text{if } b > d. \end{cases}$$

Show that $f(h,i,j,k) \in T$ and $g(a,b,c,d) \in S$.

(d) Show that the maps $f\colon S \to T$ and $g\colon T \to S$ are inverse to each other.

(e) Deduce that $\sum_{k=1}^{n} k^3 = \binom{n+1}{2}^2$.