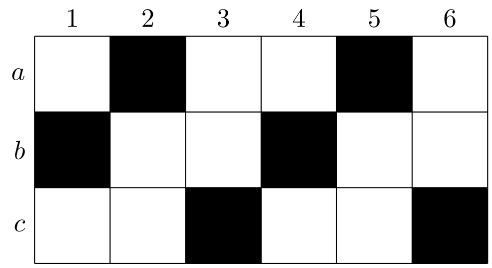
MAS334 COMBINATORICS — PROBLEM SHEET 5

Please hand in exercises 5.1 and 5.7 by the end of Week 11.

Exercise 5.1. Consider the following team allocation problem, in which each job needs a team of two people.



(a) Find $|C_U|$ for $U \subseteq \{a, b, c\}$, and thus check that the team allocation problem is plausible.
(b) Find an explicit solution.

Exercise 5.2. Which of the following are possible scores in a tournament of 8 people?
(i) $6, 6, 5, 5, 3, 2, 1, 1$.
(ii) $6, 6, 6, 5, 2, 2, 1, 0$.
(iii) $6, 6, 5, 5, 3, 2, 1, 0$.

Exercise 5.3. In a tournament of n players, let the score of player i be w_i. Let l_i denote the number of games lost by player i.
(a) Give a formula for l_i (in terms of n and w_i).
(b) Show that $l_1, \ldots, l_n$ are the scores of a tournament.

Exercise 5.4. By a *trio* we mean a tournament of three players. If we choose three players from a larger tournament and just consider the games that they play against each other, that gives a trio. A *clear winner* in a trio is a player who beats both the other players. A *clear trio* is a trio that has a clear winner. A *cyclic trio* is a trio in which the players can be labelled a, b and c such that a beats b and b beats c and c beats a.
(a) Show that every trio is either clear or cyclic. What are the score sequences for these two cases?
(b) In a tournament of n players, let w_i be the score of player i. Show that the number of trios in which player i is the clear winner is $\binom{w_i}{2}$.
(c) Deduce that the number of cyclic trios is

$$\binom{n}{3} - \sum_{k=1}^{n} \binom{w_k}{2}.$$

Exercise 5.5. This question concerns tournaments of n players $\{p_1, p_2, \ldots, p_n\}$.
(a) How many different sets of scores are there of the games so that p_1's final score is greater than p_2's which is greater than p_3's $\ldots$ which is greater than p_n's? When you have listed their scores also work out the result of each game.
(b) How many different sets of results are there of the games so that two of the players have the same score but, apart from that, all the scores are different?

(c) You are given that p_1 has the highest score and that all the other players tie second. Show that p_1 must win all their games and that n must be even.

Exercise 5.6. Consider the following Latin rectangle:

$$\begin{bmatrix} 1 & 2 & 3 & \dots & n \\ n & 1 & 2 & \dots & n-1 \end{bmatrix}$$

In how many ways can it be extended to a $3 \times n$ Latin rectangle with entries from $\{1, 2, \dots, n\}$? (You can convert this to a rook placement problem as in Problem 8.9. You should find that the relevant board is one that we have already discussed.)

Exercise 5.7. Consider the following square, which has the variable x in the bottom right corner:

$$L = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 1 & 2 \\ 3 & 4 & 6 & 1 \\ 4 & 1 & 2 & x \end{bmatrix}$$

(a) For which values of x can L be extended to a 7×7 Latin square with entries $\{1, \dots, 7\}$?
(b) For which values of x can L be extended to a 6×6 Latin square with entries $\{1, \dots, 6\}$?
(c) For the value of x in (b), find one extension of the specified type.

Exercise 5.8. Let p, q and n be positive integers with $p \leq n$ and $q \leq n$. Let L be a $p \times q$ Latin rectangle in which each of the numbers $\{1, 2, \dots, n\}$ occurs the same number of times. Show that L can be extended to an $n \times n$ Latin square.

Exercise 5.9. Write down two orthogonal 3×3 Latin squares.

Exercise 5.10. Given integers v and k with $1 < k < v$ show that there exists a design with parameters $\left(v, \binom{v}{k}, \binom{v-1}{k-1}, k, \binom{v-2}{k-2}\right)$. What are the parameters in the special case where $v > 2$ and $k = v - 1$?

Exercise 5.11. Suppose we have a design with parameters (v, b, r, k, λ). Prove that there is also a design with parameters $(v, b, b - r, v - k, b - 2r + \lambda)$.

[Hint: the idea is to replace each block by its complement in the set of varieties. You need to check that this does result in a design with the specified parameters.]

Exercise 5.12.

(a) Explain briefly how to construct a $(23, 23, 11, 11, 5)$ design.
(b) Show that, if a $(23, 23, r, k, \lambda)$ design exists, then $r = k$ and $k(k-1) = 22\lambda$. Hence find all values of r, k and λ such that a $(23, 23, r, k, \lambda)$ design exists. (Remember that to be sure that one does exist you must explain briefly how to construct it. Exercises 5.10 and 5.11 will be useful for this.)

Exercise 5.13.

(a) Show that there cannot be a design with $k = 3$, $\lambda = 1$ and $v = 11$.
(b) Show that if a design has $k = 3$ and $\lambda = 1$, then v must be congruent to 1 or 3 mod 6.

Exercise 5.14. Consider an odd number $n = 2m+1$ with $m \geq 2$. (You could take $n = 7$ to make the problem more concrete.) We could try to define a block design by taking $B = V = \mathbb{Z}/n$ and $C_j = \{j+1, \ldots, j+m\}$ for all j (where the additions are all done modulo n). Explain why this does not actually give a block design.