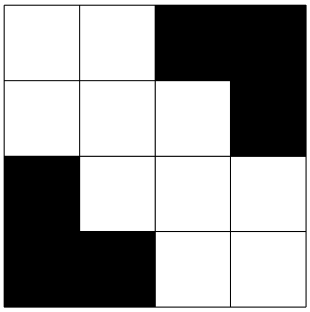
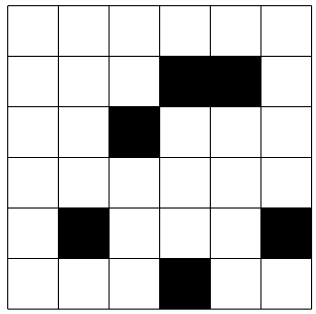
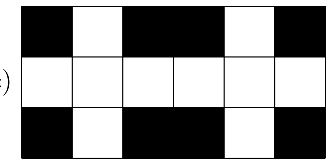
MAS334 COMBINATORICS — PROBLEM SHEET 2 — **Solutions**

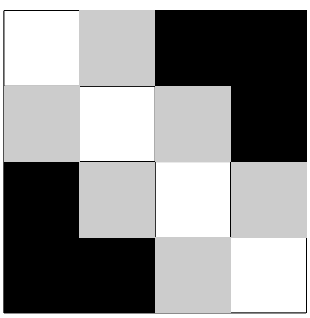
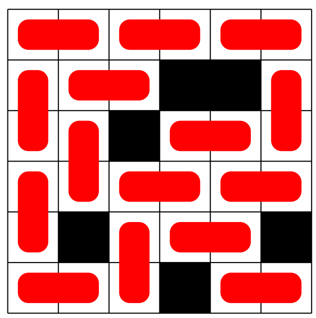
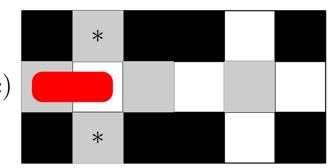
Please hand in exercises 2.1 and 2.5 by the end of Week 4.

Exercise 2.1. Consider the following boards:

(a)  (b)  (c) 

(The black squares do not count as part of the boards.) Which of these can be covered by disjoint dominos?

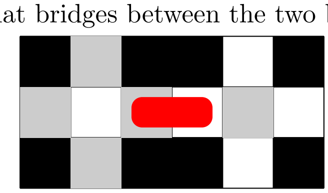
Solution: Consider the following pictures:

(a)  (b)  (c) 

We have coloured board (a) in the standard way, so every domino covers one white square and one grey square. There are 4 white squares and 6 grey squares, so no set of disjoint dominos can cover the whole of board (a). There are many ways of covering board (b), of which we have shown just one. Board (c) has 5 white squares and 5 grey squares, so we might be tempted to guess that it can also be covered by disjoint dominos, but that guess would not be correct. The only way to cover the leftmost square is to place a domino as shown above. Having done that, there is clearly no way to cover the two starred squares.

Feedback: Note that in (b), it is not enough to say that the number of white squares is the same as the number of grey squares. As board (c) shows, it is not always possible to cover the board even if the numbers are the same. For board (b), you need to actually draw the dominos to prove that covering is possible.

Also, some students wanted to analyse board (c) by dividing it into two 3×3 boards, and showing that neither of those boards can be covered. This could be part of a correct analysis, but you also need to discuss what happens if you place a domino that bridges between the two boards, as shown below.



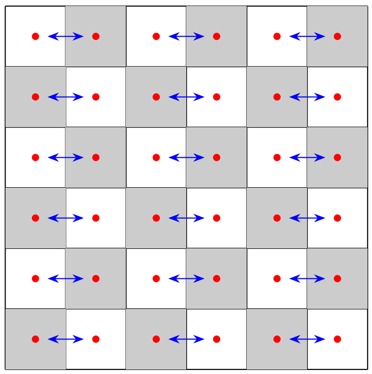
Exercise 2.2. On an $n \times n$ board there are n^2 chess pieces, one on each square. I wish to move each piece to an adjacent square (in the same row or column) so that after all the n^2 pieces have moved there is still one piece on each square.

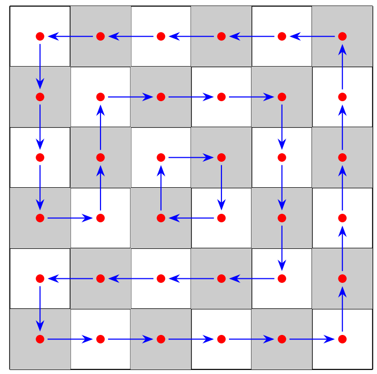
(a) Show that this can be done if n is even. (Give explicit instructions for where to move each piece, not just an abstract argument to suggest that it is possible.)

(b) By a colourful argument show that it cannot be done if n is odd. (It is not enough to try some plausible approach and show that that approach fails. There might be some complicated and nonobvious pattern of movement that does the job. You need to give a proper proof to show that that cannot happen.)

Solution:

(a) In the cases where n is even it is easy to see that such moves are possible. For example, we can swap the piece in position $(i, 2j-1)$ with the one in position $(i, 2j)$, for $i = 1, \ldots, n$ and $j = 1, \ldots, n/2$. This is shown on the left below, for the case $n = 6$. However, there are also many other possibilities, such as the one shown on the right below.





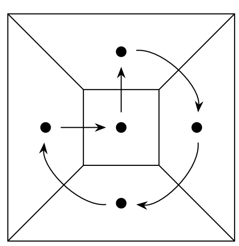
(b) Now consider the case of an $n \times n$ board, where n is odd. Picture the board chequered with black and white squares in the usual chess-board fashion. As n is odd there will be more white squares than black, say (with $\frac{1}{2}(n^2+1)$ white and $\frac{1}{2}(n^2-1)$ black). Now each move to an adjacent square is from a black square to a white or *vice-versa*. But there are more white squares than black so not all the pieces moving from the white squares will fit onto the black ones. So such moves are impossible in the odd case.

Feedback:

(a) Note that a correct answer to this question **must** include an actual description of how to move the pieces. It is not enough to give some vague argument about why it should be possible.

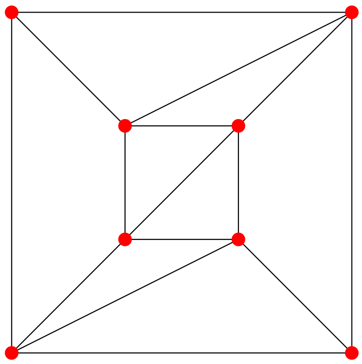
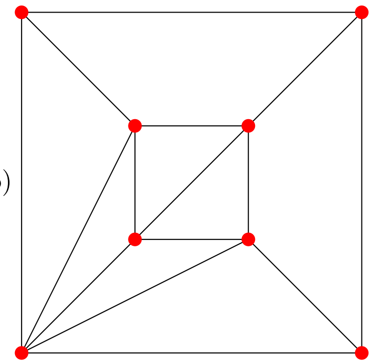
(b) Note that we need to consider the possibility of complicated patterns like the one shown on the right above, not just patterns where we swap adjacent pieces in pairs. Any proof in terms of dominos and swapping adjacent pairs is incomplete.

Note also the example of the following distorted board (which cannot be coloured black and white in the usual way):



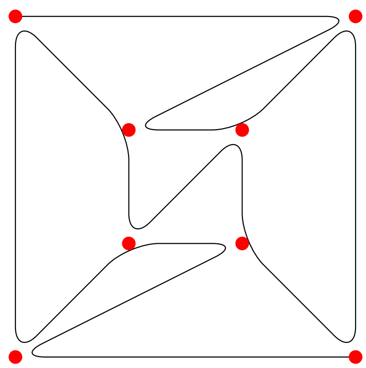
The number of squares is five, which is odd, but it is still possible to move each piece to an adjacent square. This shows that any correct proof must use the colouring of squares, not just the fact that the number of squares is odd. Also, it is not enough to give a solution for the even case, and show that essentially the same solution does not work for the odd case. Just because one method fails for the odd case, we cannot conclude that every possible method fails.

Exercise 2.3. Consider the following networks of nodes and bridges:

(a)  (b) 

In each case, say whether it is possible to have a circular tour that crosses each bridge precisely once, and also whether it is possible to have a non-circular tour that crosses each bridge precisely once.

Solution: In network (a), the top left and bottom right nodes have three bridges, and all other nodes have four bridges. According to Remark 3.6, there cannot be a circular tour, but there might be a non-circular one. In fact, the following picture shows a non-circular tour.



In network (b), all four of the outer nodes have an odd number of bridges, so we have no tour of either type.

Exercise 2.4. Given any positive integer $n \geq 1$, if we divide by two repeatedly we will eventually get to an odd number. Thus, we have $n = 2^t m$ for some $t \geq 0$ and some odd number m. We call m the *odd part* of n. For example, $60 = 2^2 \times 15$, so the odd part of 60 is 15.

Show that, given any $n + 1$ different positive integers less than or equal to $2n$, there will exist two with the same odd part. Deduce that one of those numbers is a multiple of the other.

Solution: Suppose that $a_1, \ldots, a_{n+1}$ are numbers in $[1, 2n]$, with all of them being different. Let b_i be the odd part of a_i. This is odd and less than or equal to $2n$, so it lies in the set

$$B = \{1, 3, 5, \ldots, 2n - 1\} = \{2i - 1 \mid i = 1, \ldots, n\},$$

which has $|B| = n$. As the sequence $b_1, \ldots, b_{n+1}$ is longer than the size of B, there must be a repeat, say $b_i = b_j = c$ for some $i < j$. Thus, a_i and a_j have the same odd part. Note also that $a_i = 2^r c$ and $a_j = 2^s c$ for some $r, s \geq 0$. If $r \geq s$ then a_i is a multiple of a_j, and if $r \leq s$ then a_j is a multiple of a_i.

Exercise 2.5. Let n be a positive integer. By considering numbers of the form $1, 11, 111, \ldots$ and their remainders modulo n, show that there exists a number of the form $11 \ldots 10 \ldots 00$ which is a multiple of n.

Solution: Put $a_m = 11 \cdots 1$ (with m digits), or equivalently $a_m = \sum_{i=0}^{m-1} 10^i = (10^m - 1)/9$. Fix $n > 1$, and put $\overline{a}_m = a_m \pmod{n}$, so $\overline{a}_m$ lies in the set $N = \{0, 1, \ldots, n - 1\}$. As $|N| = n$ we see that the numbers $\overline{a}_1, \ldots, \overline{a}_{n+1}$ cannot all be different. There must therefore exist indices p, q with $1 \leq p < q \leq n + 1$ with $\overline{a}_q = \overline{a}_p$. This means that the number $b = a_q - a_p$ is divisible by n. Moreover, b has the form $1 \cdots 10 \cdots 0$, with $q - p$ ones followed by p zeros.

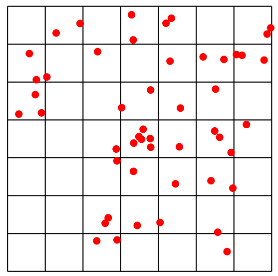
Feedback: It is important to distinguish clearly between the numbers $\overline{a}_m$ (which lie in $\{0, \ldots, n - 1\}$, but which do not have any special pattern of digits) and the numbers a_m (which have a simple pattern of digits, but do not usually lie in $\{0, \ldots, n - 1\}$).

Exercise 2.6. Prove that there exist two different powers of 7 whose difference is divisible by 1000.

Solution: Put $a_k = 7^k \pmod{1000}$, so $a_k \in [0, 1000)$. The set $[0, 1000)$ only has 1000 elements, so the numbers $a_1, \ldots, a_{1001}$ cannot all be different, so we can find $i < j$ with $a_i = a_j$. This means that $7^j = 7^i \pmod{1000}$, so $7^j - 7^i$ is divisible by 1000. (In fact, one can check directly that $7^{20} = 1 \pmod{1000}$, and so $7^{i+20} - 7^i$ is divisible by 1000 for all i.)

Exercise 2.7. Let $a_1, \ldots, a_{50}$ be points in the unit square $[0, 1]^2$. Show that there are indices $i < j$ such that the distance from a_i to a_j is less than 0.21. (Hint: divide the square into small boxes and apply the pigeonhole principle.)

Solution: For $u, v \in \{1, \ldots, 7\}$, put $Q_{uv} = [(u-1)/7, u/7] \times [(v-1)/7, v/7]$. This gives a grid of $7^2 = 49$ small squares with sides of length $1/7$, and together they cover the whole of the unit square.



Because we have 50 points and 49 boxes, there must be some box Q_{uv} that contains at least two of our points, so we can choose $i < j$ with $a_i, a_j \in Q_{uv}$. It is clear that the distance between a_i and a_j is at most as large as the distance between opposite corners of Q_{uv}. As the sides of Q_{uv} have length $1/7$, we see that the distance between opposite corners is $\sqrt{2}/7 \simeq 0.202 < 0.21$.

Exercise 2.8. Using the pigeonhole principle, explain why there is no compression algorithm that can compress every possible $8MB$ file down to $1MB$ in such a way that it can be uncompressed without errors.

Solution: Let $f_1, \ldots, f_n$ be the list of all possible $8MB$ files, and let m be the number of possible $1MB$ files. It is clear that n is much bigger than m. In fact, an $8MB$ file can be regarded as a sequence of 8 files of size $1MB$, which shows that $n = m^8$. If we want precise numbers, we can recall that $1MB$ is officially 2^{20} bytes or $2^{8\times 20} = 2^{160} \simeq 1.46 \times 10^{48}$ bits, so $m = 2^{2^{160}}$, which is ridiculously enormous. We then have $n = m^8 = 2^{8\times 2^{160}} = 2^{2^{163}}$, which is even more ridiculously enormous. Now suppose we have a compression algorithm, which converts each $8MB$ file to a $1MB$ file. Let $\overline{f}_i$ be the compressed form of f_i. There are only m possibilities for $\overline{f}_i$, so the files $\overline{f}_1, \ldots, \overline{f}_n$ cannot all be different. There must therefore exist indices $i < j$ and a $1MB$ file g such that $\overline{f}_i = \overline{f}_j = g$. If the uncompression process works correctly then it must convert g to f_i, and it must also convert g to f_j, but this is impossible because $f_i \neq f_j$.

Of course, normal human activity produces computer files that have a lot of non-random structure, and it is possible to design compression algorithms that work well on such files. However, for every such algorithm, it is possible to come up with strangely random files that actually get bigger when you try to compress them.

Exercise 2.9. Lemma 5.4 says that for a finite, nonempty set I we have $\sum_{J\subseteq I}(-1)^{|J|} = 0$. Check this explicitly in the case $I = \{a, b, c\}$.

Solution: We can list the subsets $J \subseteq I$, the sizes $|J|$ and the terms $(-1)^{|J|}$ as follows:

J	$\emptyset$	a	b	c	ab	ac	bc	abc
$\lvert J\rvert$	0	1	1	1	2	2	2	3
$(-1)^{\lvert J\rvert}$	1	-1	-1	-1	1	1	1	-1

This gives

$$\sum_{J\subseteq I}(-1)^{|J|} = 1 + (-1) + (-1) + (-1) + 1 + 1 + 1 + (-1) = 0,$$

as expected.

Exercise 2.10. Suppose that we choose a number $k \in \{0, 1, \dots, 999\}$ at random. What is the probability that k and 1000 are coprime?

Solution: The primes dividing 1000 are 2 and 5. Thus, Proposition 5.13 tells us that the answer is

$$x = (1 - \tfrac{1}{2})(1 - \tfrac{1}{5}) = \tfrac{1}{2} \times \tfrac{4}{5} = 0.4.$$

Exercise 2.11. Suppose we have a set B with subsets $B_1, \dots, B_4$ such that $B = B_1 \cup \cdots \cup B_4$ and $|B|$ is odd. Suppose there are numbers n_1, n_2, n_3 such that

- $|B_i| = n_1$ for all i
- $|B_{ij}| = n_2$ for all $i < j$
- $|B_{ijk}| = n_3$ for all $i < j < k$.

Prove that $|B_{1234}|$ is odd.

Solution: The positive form of the IEP gives

$$|B| = |B_1 \cup B_2 \cup B_3 \cup B_4| = \sum_i |B_i| - \sum_{i<j} |B_{ij}| + \sum_{i<j<k} |B_{ijk}| - |B_{1234}|.$$

On the right hand side, the first sum has 4 terms all equal to n_1. The second sum has $\binom{4}{2} = 6$ terms, all equal to n_2. The third sum has $\binom{4}{3} = 4$ terms, all equal to n_3. The equation therefore becomes

$$|B| = 4n_1 - 6n_2 + 4n_3 - |B_{1234}|.$$

From this it is clear that $|B_{1234}| = |B| \pmod 2$. We are given that $|B|$ is odd, and it follows that $|B_{1234}|$ is also odd.

Exercise 2.12. Consider the set $X = [2, 1000] = \{2, 3, 4, \dots, 1000\}$. For each prime p, let X_p be the subset of numbers in X that are divisible by p. As in the Inclusion-Exclusion Principle, we write $X_{p,q,r}$ for $X_p \cap X_q \cap X_r$, and so on. We will investigate the sizes of some sets related to these. It will be helpful to use the notation

$$\lfloor x \rfloor = \text{ largest integer } n \text{ such that } n \leq x$$

(so for example $\lfloor 7.01 \rfloor = \lfloor 7.89 \rfloor = \lfloor 7.77 \rfloor = 7$).

(a) Show that $|X_2| = 5000$ and $|X_{2,3}| = 1666$.
(b) Give a formula for $|X_{p,q,r}|$.
(c) Using the IEP, show that there are precisely 7334 numbers in X that are divisible by at least one of the primes 2, 3 and 5. Of these 7334 numbers, show that 3 are prime and 7331 are not prime.
(d) Give an upper bound for the number of primes in X. How would this bound change if we tested for divisibility by 7 as well as 2, 3 and 5?

Solution:

(a) First, we have

$$X_2 = \{2, 4, 6, \dots, 1000\} = \{2k \mid 1 \leq k \leq 5000\},$$

so $|X_2| = 5000$. Next, note that a number lies in $X_{2,3}$ iff it is divisible by both 2 and 3, iff it is divisible by 6. Moreover, we have $6k \leq 10000$ iff

$$k \leq \lfloor 10000/6 \rfloor = \lfloor 1666.666\dots \rfloor = 1666.$$

This means that $X_{2,3} = \{6k \mid 1 \leq k \leq 1666\}$, and so $|X_{2,3}| = 1666$.

(b) Similarly, $X_{p,q,r}$ is the set of numbers $x \in X$ that are divisible by p, q and r. As p, q and r are distinct primes, we see that x is divisible by p, q and r iff it is divisible by the product pqr. This means that

$$X_{p,q,r} = \{pqrk \mid 1 \leq k \leq \lfloor 10000/(pqr) \rfloor\},$$

and so $|X_{p,q,r}| = \lfloor 10000/(pqr) \rfloor$.

(c) We are asked to find $|Y|$, where $Y = X_2 \cup X_3 \cup X_5$. By the IEP, this is given by

$$\begin{aligned}
|Y| &= |X_2| + |X_3| + |X_5| - |X_{2,3}| - |X_{2,5}| - |X_{3,5}| + |X_{2,3,5}| \\
&= \left\lfloor \frac{10^4}{2} \right\rfloor + \left\lfloor \frac{10^4}{3} \right\rfloor + \left\lfloor \frac{10^4}{5} \right\rfloor - \left\lfloor \frac{10^4}{2 \times 3} \right\rfloor - \left\lfloor \frac{10^4}{2 \times 5} \right\rfloor - \left\lfloor \frac{10^4}{3 \times 5} \right\rfloor + \left\lfloor \frac{10^4}{2 \times 3 \times 5} \right\rfloor \\
&= 5000 + 3333 + 2000 - 1666 - 1000 - 666 + 333 = 7334.
\end{aligned}$$

Note that Y includes the numbers 2, 3 and 5 themselves, which are prime. Each of the remaining 7331 elements of Y has the form $2k$ or $3k$ or $5k$ with $k > 1$, and so is visibly not prime.

(d) We have seen that there are at least 7331 non-prime elements of X, and $|X| = 9999$, so there are at most $9999 - 7331 = 2668$ primes in X. To get a slightly better estimate, we can consider the set $Z = X_2 \cup X_3 \cup X_5 \cup X_7$. By the same method as above, we find that $|Z| = 7715$. Moreover, the numbers $2, 3, 5, 7$ are the only primes in Z, so we have at least 7711 non-primes. This means that there are at most $9999 - 7711 = 2288$ primes in X. (In fact, the actual number of primes in X is 1229.)

Exercise 2.13. Let A and B be finite sets, with $|A| = m$ and $|B| = n$ say. We will assume that $m \geq n$. Let F be the set of all functions from A to B. For each $b \in B$, let $F_b \subseteq F$ be the subset of functions $f\colon A \to B$ such that $b \notin f(A)$. We also let $E \subseteq F$ be the set of surjective functions.

(a) Explain why $|F| = n^m$.
(b) Show that $|F_b| = (n-1)^m$ for all $b \in B$.
(c) Show that for $b \neq c$ in B we have $|F_b \cap F_c| = (n-2)^m$.
(d) Give a formula for E in terms of the sets F_b.
(e) By applying the negative IEP to your formula in (d), prove that

$$|E| = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^m.$$

Solution:

(a) List the elements of A as $a_1, \dots, a_m$. To specify a function $f\colon A \to B$, we need to choose the image $f(a_i) \in B$ for $i = 1, \dots, m$. As $|B| = n$, there are n choices for each value $f(a_i)$. This gives n^m choices in total.

(b) To choose an element $f \in F_b$, we again need to choose the values $f(a_i)$ for $i = 1, \dots, m$. To ensure that f lies in F_b, we need to make sure that $f(a_i) \neq b$ for all i. This means that each value $f(a_i)$ must lie in the set $B \setminus \{b\}$, which has size $n - 1$. We therefore get $|F_b| = (n-1)^m$.

(c) Similarly, to choose an element $f \in F_b \cap F_c$, we need to choose the values $f(a_i)$ for $i = 1, \dots, m$, and all of these must lie in the set $B \setminus \{b, c\}$, which has size $n - 2$. This gives $|F_b \cap F_c| = (n-2)^m$.

(d) A function $f\colon A \to B$ is surjective iff for every element $b \in B$ we have $b \in f(A)$, so $f \notin F_b$. Thus, we have

$$E = \{f \mid \forall b, f \notin F_b\} = \{f \mid f \notin \bigcup_b F_b\} = F \setminus \bigcup_b F_b.$$

(e) The negative form of the IEP now tells us that

$$|E| = \left| F \setminus \bigcup_b F_b \right| = \sum_{I \subseteq B} (-1)^{|I|} |F_I|.$$

Generalizing parts (b) and (c) in an obvious way, we see that if $|I| = k$ then $|F_I| = (n-k)^m$. Moreover, there are $\binom{n}{k}$ possible choices of I with $|I| = k$. This gives

$$|E| = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^m.$$

Exercise 2.14.

(a) Explain briefly why there are $\binom{n-1}{k-1}$ positive integer solutions of the equation

$$x_1 + x_2 + \cdots + x_k = n.$$

(b) Use the Inclusion/Exclusion Principle to find the number of positive integer solutions of the equation

$$x_1 + x_2 + x_3 = 20$$

satisfying the conditions $x_1 \leq 5$, $x_2 \leq 10$ and $x_3 \leq 15$.

Solution: Part (a) is in the lecture notes and will not be repeated here. For part (b), put

$$\begin{aligned}
X &= \{(x_1, x_2, x_3) \mid x_1 > 0,\ x_2 > 0,\ x_3 > 0,\ x_1 + x_2 + x_3 = 20\} \\
X_1 &= \{x \in X \mid x_1 > 5\} \\
X_2 &= \{x \in X \mid x_2 > 10\} \\
X_3 &= \{x \in X \mid x_3 > 15\} \\
Y &= \{x \in X \mid x_1 \leq 5,\ x_2 \leq 10,\ x_3 \leq 15\}.
\end{aligned}$$

We are asked to find $|Y|$. It is easy to see that $Y = X \setminus (X_1 \cup X_2 \cup X_3)$, so we can use the negative form of the IEP to get

$$|Y| = |X| - |X_1| - |X_2| - |X_3| + |X_{12}| + |X_{13}| + |X_{23}| - |X_{123}|.$$

Part (a) tells us that $|X| = \binom{19}{2} = 171$. For a solution $x \in X_{13}$ we must have $x_1 > 5$ and $x_2 > 0$ and $x_3 > 15$ and $x_1 + x_2 + x_3 = 20$, but this is clearly impossible, so $X_{13} = \emptyset$ and $|X_{13}| = 0$. The same argument shows that $X_{23} = X_{123} = \emptyset$, so the IEP equation simplifies to

$$|Y| = 171 - |X_1| - |X_2| - |X_3| + |X_{12}|.$$

To understand the remaining terms, we put $y_1 = x_1 - 5$ and $y_2 = x_2 - 10$ and $y_3 = x_3 - 15$, so that $y_i > 0$ for solutions in X_i.

- X_1 is the set of positive solutions to $y_1 + x_2 + x_3 = 15$, so $|X_1| = \binom{14}{2} = 91$.
- X_2 is the set of positive solutions to $x_1 + y_2 + x_3 = 10$, so $|X_2| = \binom{9}{2} = 36$.
- X_3 is the set of positive solutions to $x_1 + x_2 + y_3 = 5$, so $|X_3| = \binom{4}{2} = 6$.
- X_{12} is the set of positive solutions to $y_1 + y_2 + x_3 = 5$, so $|X_{12}| = \binom{4}{2} = 6$.

We now have

$$|Y| = 171 - 91 - 36 - 6 + 6 = 44.$$

Exercise 2.15. Let P be the set of permutations of $\{1, \dots, 9\}$. Let Q be the subset of permutations $\sigma \in P$ satisfying $\sigma(i) + i \leq 10$ for all i. Let R be the subset of permutations $\sigma \in P$ satisfying $\sigma(i) = i \pmod 3$ for all i. Find $|P|$, $|Q|$ and $|R|$.

Solution: It is standard that $|P| = 9! = 362880$. If $\sigma \in Q$ then we must have $\sigma(9) + 9 \leq 10$ so $\sigma(9) \leq 1$ but $\sigma(9) \in \{1, \dots, 9\}$ so $\sigma(9) = 1$. We then have $\sigma(8) + 8 \leq 10$ so $\sigma(8) \in \{1, 2\}$, but $\sigma(8)$ must be different from $\sigma(9)$, so $\sigma(8) = 2$. We then have $\sigma(7) + 7 \leq 10$ so $\sigma(7) \in \{1, 2, 3\}$, but $\sigma(7)$ must be different from $\sigma(8)$ and $\sigma(9)$, so $\sigma(7) = 3$. By proceding in the same way, we find that $\sigma(i) = 10 - i$ for all i. In particular, there is only one possible choice for σ, so $|Q| = 1$. Finally, any permutation $\sigma \in R$ must permute the sets $N_1 = \{1, 4, 7\}$, $N_2 = \{2, 5, 8\}$ and $N_3\{3, 6, 9\}$ separately. There are six possible choices for the effect of σ on N_1, and six possible choices for the effect of σ on N_2, and six possible choices for the effect of σ on N_3. This gives $|R| = 6^3 = 216$.

Exercise 2.16. Put $B = \{1, \dots, n\}$ and $B_i = \{1, \dots, i\} \subseteq B$ for $i = 1, \dots, n$. Put $B' = B_1 \cup \dots \cup B_n$. What is $|B'|$? (The answer is obvious.) Check that the IEP gives the same as the obvious answer. (Hint: group the nonempty subsets $I \subseteq \{1, \dots, n\}$ according to their minimum elements.)

Solution: As $B = B_n \subseteq B_1 \cup \dots \cup B_n = B' \subseteq B$ we see that $B' = B$ and so $|B'| = |B| = n$. On the other hand, the IEP tells us that $|B'| = \sum_{I \neq \emptyset} (-1)^{|I|+1} |B_I|$. We can group the terms as in the hint to give

$$|B'| = \sum_{m=1}^{n} \sum_{\min(I) = m} (-1)^{|I|+1} |B_I|.$$

Now suppose that $\min(I) = m$, so $I = \{m, i_1, \dots, i_r\}$ for some $i_1, \dots, i_r > m$. We then have $B_m \subset B_{i_t}$ for all t, so

$$B_I = B_m \cap B_{i_1} \cap \cdots \cap B_{i_r} = B_m,$$

so $|B_I| = |B_m| = m$. The IEP formula now becomes

$$|B'| = \sum_{m=1}^{n} m \sum_{\min(I)=m} (-1)^{|I|+1}.$$

Now fix m and put $M = \{m+1, \dots, n\}$. Any subset I with $\min(I) = m$ can be expressed as $I = \{m\} \cup J$ with J an arbitrary subset of M. This gives $(-1)^{|I|+1} = (-1)^{|J|}$. By taking the sum over all J, we get $\sum_{\min(I)=m}(-1)^{|I|+1} = \sum_{J \subseteq M}(-1)^{|J|}$. By Lemma 5.4, this is zero unless $M = \emptyset$, which only occurs when $m = n$. Thus, in our IEP formula we can discard the terms for $m = 1, \dots, n-1$. In the case $m = n$ the only relevant set I is $I = \{n\}$ with $(-1)^{|I|+1} = 1$, so we get $|B'| = n$, which agrees with the obvious answer.

Exercise 2.17. Put $B = \{0, 1, \dots, 15\}$. Any number $k \in B$ can be expressed in base 2 with four binary digits, for example

$$11 = 8 + 2 + 1 = 1.2^3 + 0.2^2 + 1.2^1 + 1.2^0 = 1011_2.$$

Let B_i be the subset of those $k \in B$ such that the binary expansion contains 2^i. (For example, the expansion of 11 contains 2^3 but not 2^2, so $11 \in B_3$ but $11 \notin B_2$.) As usual, we put $B^* = B \setminus (B_0 \cup B_1 \cup B_2 \cup B_3)$. What is $|B^*|$? (The answer is easy.) Check that the IEP gives the same as the easy answer.

Solution: The set B^* consists of those $k \in B$ such that the binary expansion of k does not contain 2^0, 2^1, 2^2 or 2^3, which is only possible for $k = 0$. We thus have $B^* = \{0\}$ and $|B^*| = 1$.

On the other hand, the IEP gives

$$|B^*| = \sum_{I \subseteq \{0,1,2,3\}} (-1)^{|I|} |B_I|.$$

Now consider a subset $I \subseteq \{0, 1, 2, 3\}$, with $|I| = r$ say. For a number k to lie in B_I, it must have a 1 in the positions corresponding to the elements of I, but in the other $4 - r$ positions it can have either a 0 or a 1. This gives $|B_I| = 2^{4-r}$. The number of possible choices of I with $|I| = r$ is $\binom{4}{r}$. If we use this to rewrite the IEP formula and then use the binomial expansion we get

$$|B^*| = \sum_{r=0}^{4} \binom{4}{r} (-1)^r 2^{4-r} = (-1+2)^4 = 1^4 = 1,$$

which agrees with our previous approach.