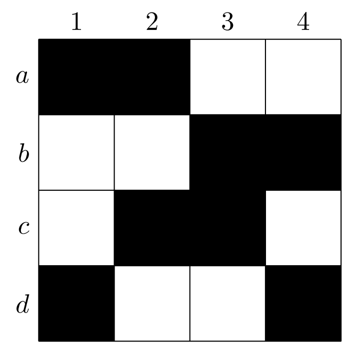
MAS334 COMBINATORICS — PROBLEM SHEET 3 — Solutions

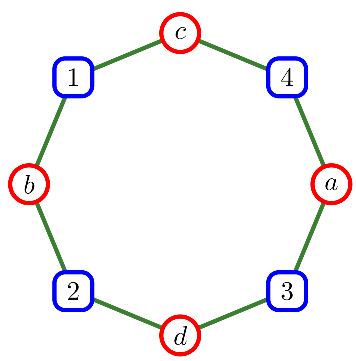
Please hand in exercises 3.1 and 3.2 by the end of Week 6.

Exercise 3.1. Consider the following board:



(a) Draw the corresponding incidence graph.
(b) Write down the row sets R_x and column sets C_y.
(c) There are two possible full matchings. Draw the corresponding rook placements and incidence graphs as in Example 6.7.

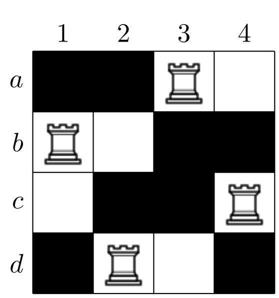
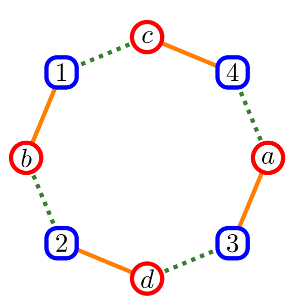
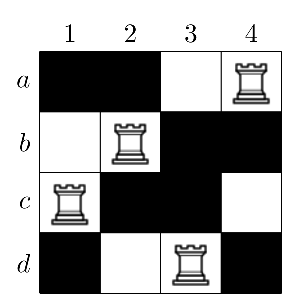
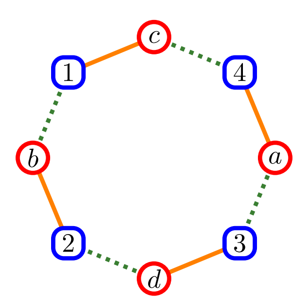
Solution: The incidence graph is as follows:



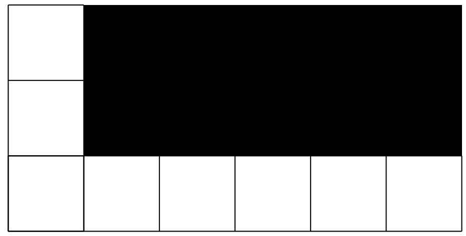
The row and column sets are as follows:

$$R_a = \{3,4\} \qquad R_b = \{1,2\} \qquad R_c = \{1,4\} \qquad R_d = \{2,3\}$$
$$C_1 = \{b,c\} \qquad C_2 = \{b,d\} \qquad C_3 = \{a,d\} \qquad C_4 = \{a,c\}.$$

The two full matchings are $a3b1c4d2$ and $a4b2c1d3$; they can be drawn as follows.

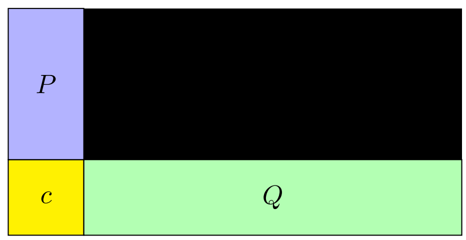





Exercise 3.2. Consider an L-shaped board B, with p squares on the left edge and q squares on the bottom edge. The case $(p, q) = (3, 6)$ is illustrated below.



What is the rook polynomial? (Give a formula for all p and q.)

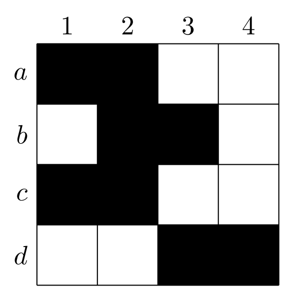
Solution: Let c denote the bottom left corner. Let P be the left edge excluding c, so $|P| = p - 1$. Let Q be the bottom edge excluding c, so $|Q| = q - 1$.



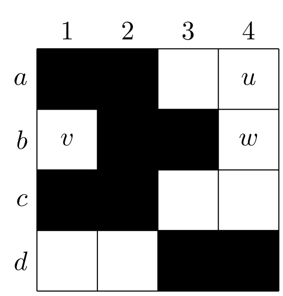
We then have $B = P \cup Q \cup \{c\}$ and $c_1(B) = |B| = (p-1) + (q-1) + 1 = p + q - 1$. The only way we can place two rooks is to have one in P and the other in Q, so $c_2(B) = |P||Q| = (p-1)(q-1)$. Once we have placed two rooks, there is no way to place any more, so $c_k(B) = 0$ for $k \geq 3$. We therefore have

$$r_B(x) = 1 + (p + q - 1)x + (p-1)(q-1)x^2.$$

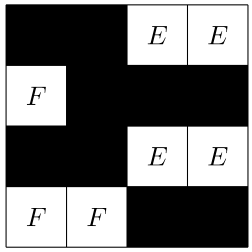
Exercise 3.3. Find the rook polynomial of the following board B:

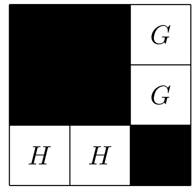


Solution: This can be done using the blocking/stripping/factoring method, or by just tabulating the solutions. If we use blocking and stripping, it makes a big difference which square we start with. Some choices give a very efficient calculation but other choices require much more work, so we should choose carefully. Any choice of square will give us two new boards to consider. It will be good if at least one of those boards can be factored as in Theorem 8.7, and even better if both of the new boards can be split. The most popular choices are the squares marked u, v and w:



Of these choices, w is the best, v is almost as good, and u is not so good. If we use w then we get $r_B(x) = r_C(x) + x\, r_D(x)$, where

$C =$ 

$D =$ 

Here we have divided C into two fully disjoint subboards E and F, and we have divided D into two fully disjoint subboards G and H. It is easy to find $r_E(x), \dots, r_H(x)$ directly, and then we can use factoring to get $r_C(x)$ and $r_D(x)$, and blocking and stripping to get $r_B(x)$:

$$r_E(x) = 1 + 4x + 2x^2 \qquad r_F(x) = 1 + 3x + x^2$$
$$r_G(x) = 1 + 2x \qquad r_H(x) = 1 + 2x$$

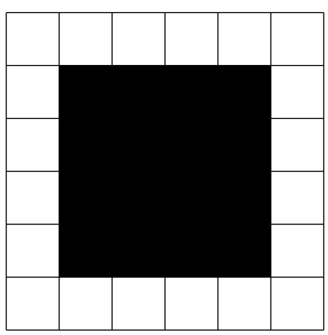
$$\begin{aligned} r_C(x) &= r_E(x) r_F(x) = 1 + 7x + 15x^2 + 10x^3 + 2x^4 \\ r_D(x) &= r_G(x) r_H(x) = 1 + 4x + 4x^2 \\ r_B(x) &= r_C(x) + x\, r_D(x) = 1 + 8x + 19x^2 + 14x^3 + 2x^4. \end{aligned}$$

We can also just tabulate the solutions. To avoid mistakes, I strongly recommend writing everything in dictionary order. We get the following table:

- 1 rook: $a3$, $a4$, $b1$, $b4$, $c4$, $c4$, $d1$, $d2$ (8 possibilities)
- 2 rooks: $a3b1$, $a3b4$, $a3c4$, $a3d1$, $a3d2$, $a4b1$, $a4c3$, $a4d1$, $a4d2$, $b1c3$, $b1c4$, $b1d2$, $b4c3$, $b4d1$, $b4d2$, $c3d1$, $c3d2$, $c4d1$, $c4d2$ (19 possibilities)
- 3 rooks: $a3b1c4$, $a3b1d2$, $a3b4d1$, $a3b4d2$, $a3c4d1$, $a3c4d2$, $a4b1c3$, $a4b1d2$, $a4c3d1$, $a4c3d2$, $b1c3d2$, $b1c4d2$, $b4c3d1$, $b4c3d2$ (14 possibilities)
- 4 rooks: $a3b1c4d2$, $a4b1c3d2$ (2 possibilities).

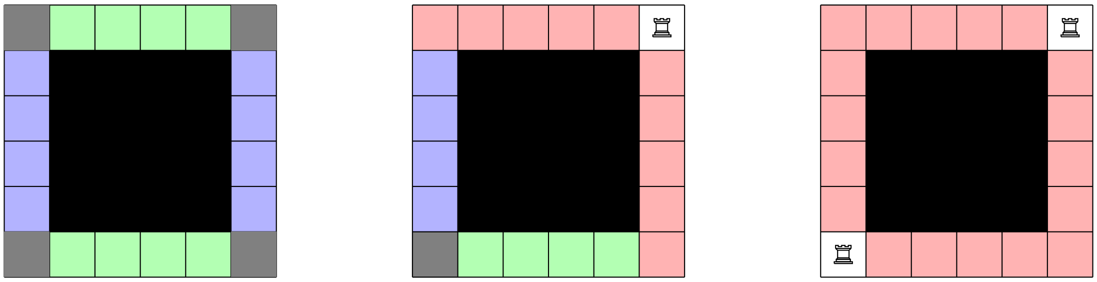
This again gives $r_B(x) = 1 + 8x + 19x^2 + 14x^3 + 2x^4$.

Exercise 3.4. Consider a ring-shaped board R_n of size $n \times n$ with $n \geq 3$, as illustrated below for the case $n = 6$. What is the rook polynomial?



It is best to start by thinking about rooks in the corner positions, and then think about what else we can do after the corner rooks are in place. The answer comes out as a moderately complicated expression, which can be expanded and simplified. If you wish, you can ignore the expansion and simplification step, or get a computer to do it.

Solution: We will count the placements separately according to the number of rooks that we have on the corner positions. If there are no corner rooks, then we are effectively placing rooks on the board shown on the left below, where the corners have been blocked off.



This board can be factored into two copies of the full board of shape $2 \times (n-2)$, which has rook polynomial

$$f(x) = 1 + 2(n-2)x + (n-2)(n-3)x^2,$$

as we see from Proposition 7.11. Thus, the left hand board has polynomial $f(x)^2$.

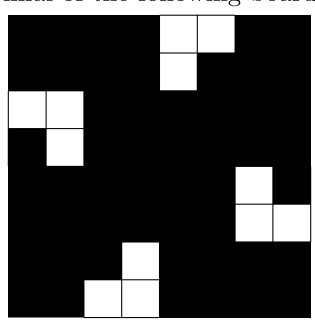
We next consider placements with precisely one corner rook, as in the middle picture. The available space for further rooks then factors as two full boards of shape $1 \times (n-2)$, which have rook polynomial $g(x) = 1 + (n-2)x$. We need to multiply by x to account for the corner rook, and then multiply by 4 to account for the fact that there are four possible corners. Thus, the contribution of placements with one corner rook is $4x\, g(x)^2$. Next, if we want two corner rooks, then they have to go in opposite corners. There are two choices for which pair of corners we use, and no space for futher rooks anywhere else. Thus, placements of this type contribute $2x^2$ to the rook polynomial. We conclude that

$$r_{R_n}(x) = f(x)^2 + 4x\, g(x)^2 + 2x^2.$$

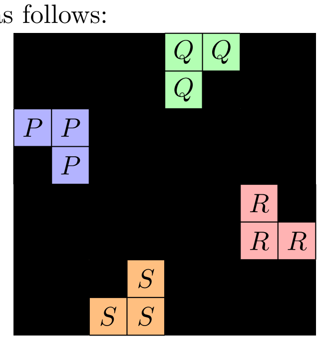
This can be expanded and simplified to give

$$r_{R_n}(x) = 1 + 4(n-1)x + (6n^2 - 18n + 14)x^2 + 4(n-2)^3x^3 + (n-2)^2(n-3)^2x^4.$$

Exercise 3.5. What is the rook polynomial of the following board?



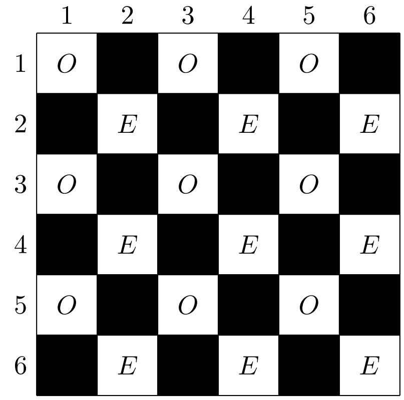
Solution: The board can be factored as follows:



Any two labelled squares in the same row have the same label, and any two labelled squares in the same column have the same label, so the factoring theorem is applicable. Each of the four pieces has rook polynomial $1 + 3x + x^2$, by inspection. The rook polynomial of the full board is therefore $(1 + 3x + x^2)^4$.

Exercise 3.6. Let n be a positive even integer and consider an $n \times n$ chess-board in which the squares are coloured black and white in the usual chequered fashion. In how many ways can n non-challenging rooks be placed on the white squares?

Solution: The case $n = 6$ looks like this:



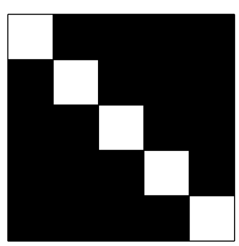
For half of the white squares, the row and column numbers are both odd. We have marked these squares O. For the other half of the white squares, the row and column numbers are both even. We have marked these squares E. Although we have only drawn the case $n = 6$, it works the same way whenever n is positive and even. Our board therefore splits as a disjoint union of two empty boards of size $(n/2) \times (n/2)$. There are $(n/2)!$ ways of placing a full set of non-challenging rooks on the O board, and the same number for the E board, so there are $(n/2)!^2$ ways of placing n non-challenging rooks on the white squares of the full board.

Exercise 3.7. Which of the following polynomials can be the rook polynomial of a board? Give reasons, including examples of appropriate boards, where possible.

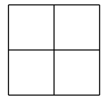
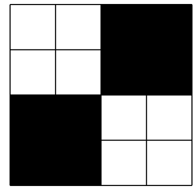
$$\begin{aligned}
p_1(x) &= 1 + x \\
p_2(x) &= (1 + x)^n \\
p_3(x) &= 1 + 4x + 2x^2 \\
p_4(x) &= (1 + 4x + 2x^2)^2 \\
p_5(x) &= 1 - 3x \\
p_6(x) &= 1 + 2x + 2x^2.
\end{aligned}$$

Solution:

- The polynomial $p_1(x) = 1 + x$ is the rook polynomial of the full 1×1 board F_1.
- The polynomial $p_2(x) = (1 + x)^n$ is the rook polynomial of the diagonal board D_n, in which the diagonal squares are white and all other squares are black. For example, $(1 + x)^5$ is the rook polynomial of the following board:

  This follows easily from the factoring theorem (Theorem 8.7), because we have a disjoint union of n copies of the full 1×1 board.
- The polynomial $p_3(x) = 1 + 4x + 2x^2$ is the rook polynomial of the full 2×2 board, as shown on the left below. So the factoring theorem tells us that $p_4(x) = (1 + 4x + 2x^2)^2$ is the rook polynomial of the disjoint union of two copies of the full 2×2 board, as shown on the right below.

- The polynomial $p_5(x) = 1 - 3x$ is not a rook polynomial: the coefficients in a rook polynomial are always non-negative integers, since they are the number of ways of placing non-challenging rooks. A coefficient -3 is not possible.
- The polynomial $p_6(x) = 1 + 2x + 2x^2$ is also not a rook polynomial. Since the coefficient of x is 2, there would have to be two unshaded squares on the board. If the two squares share the same row or the same column, then there is no way of placing two non-challenging rooks, so the coefficient of x^2 would be zero. If the two squares do not share the same row or the same column, then there is precisely one way of placing two non-challenging rooks, so the coefficient of x^2 would be 1. But there is no way that the coefficient of x^2 can be 2.