MAS334 COMBINATORICS 2017/2018

Solutions to Example Sheet 1 : The binomial coefficients

1. Recall the Binomial Theorem (Theorem 6):

$$\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n = (1+x)^n.$$

The first two results required are obtained by simply substituting $x = 1$ and then $x = -1$ into this.

Then adding those first two results gives the sum of the even terms as 2^{n-1}, (since twice the sum of the even terms is 2^n).

Similarly subtracting the two results gives the sum of the odd terms as 2^{n-1} too.

In the homework, most students took the approach described above. Here are two interesting alternatives. First, we want to understand the sums

$$S = \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \sum_{k\geq 0}\binom{n}{2k}$$
$$T = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots = \sum_{k\geq 0}\binom{n}{2k+1}.$$

Pascal's relation tells us that $\binom{n}{2k+1} = \binom{n-1}{2k} + \binom{n-1}{2k+1}$, so

$$T = \sum_{k\geq 0}\binom{n-1}{2k} + \sum_{k\geq 0}\binom{n-1}{2k+1} = \sum_{\text{even } j\geq 0}\binom{n-1}{j} + \sum_{\text{odd } j\geq 0}\binom{n-1}{j} = \sum_{j\geq 0}\binom{n-1}{j} = 2^{n-1}.$$

By subtracting this from the relation $S+T = \sum_{j\geq 0}\binom{n}{j} = 2^n$, we also get the relation $S = 2^{n-1}$.

So far we have just given algebraic proofs. For a combinatorial approach:

- Let E be the collection of subsets $A \subseteq \{1, \ldots, n\}$ such that $|A|$ is even.
- Let O be the collection of subsets $B \subseteq \{1, \ldots, n\}$ such that $|B|$ is odd
- Let P be the collection of subsets $C \subseteq \{1, \ldots, n-1\}$ of any size.

It is easy to see that $|E| = S$ and $|O| = T$ and $|P| = 2^{n-1}$, so we want to show that $|E| = |O| = |P|$, which we can do by giving a one to one correspondence between these sets.

Given a set $C \subseteq \{1, \ldots, n-1\}$ we define

$$A = \begin{cases} C & \text{if } |C| \text{ is even} \\ C \cup \{n\} & \text{if } |C| \text{ is odd} \end{cases} \qquad B = \begin{cases} C & \text{if } |C| \text{ is odd} \\ C \cup \{n\} & \text{if } |C| \text{ is even.} \end{cases}$$

It is not hard to check that $A \in E$ and $B \in O$ and that this procedure gives the required one to one correspondence between P, E and O.

2. The answer is 2^n. You could count the subsets of various sizes: there are $\binom{n}{k}$ subsets of k elements. Then adding all these up for the various k would give (by question 1) 2^n. However it is much quicker to see that a subset is determined by deciding for each of $1, 2, 3, \ldots, n$ whether it is IN the subset or OUT of the subset; i.e. a two-way choice n times, giving 2^n options overall.

3. A rectangle is determined by its sides. These are obtained by choosing any two of the $n+1$ horizontal lines and any two of the $n+1$ vertical lines. So the number of rectangles is $\binom{n+1}{2}^2$.

[You are not necessarily expected to spot this two line solution straight away. If not, you should begin by considering small examples. Work out what the answer is in the $n \times n$ case, where $n = 1, 2, 3, 4, 5$, by directly counting. Make sure you do this carefully and accurately, as mistakes here will lead to you failing to spot a pattern. You should get the answers $1, 9, 36, 100, 225$. Now look for a pattern. The first thing you notice is that these numbers are all squares: $1^2, 3^2, 6^2, 10^2, 15^2$. Now you need to see a pattern in the numbers $1, 3, 6, 10, 15$. These you should spot in Pascal's triangle, as $\binom{n+1}{2}$ for $n = 1, 2, 3, 4, 5$. So at this point you *guess* that the general answer is $\binom{n+1}{2}^2$. You still need to find an argument to justify this guess in the general case. But now you know that you need to look in the problem for a choice of two things from $n+1$ things, twice over. Now you should be able to find the two line argument above.]

In the homework, many students gave some version of the above proof, with varying degrees of coherence. Many other students guessed the correct formula, but did not give a proof. To get the right formula is definitely a step forward and is a worthwhile partial answer. However, you should be clear about what you have proved and what you have not proved. You could say: "we have checked that the number of rectangles is $\binom{n}{2}^2$ for $n = 1, \ldots, 5$, and we conjecture that this formula is valid for all n". You should not say "therefore the answer is $\binom{n}{2}^2$ for all n" unless you have actually given a proof.

A few students also guessed (with reasonable evidence) that the number of rectangles is $\sum_{k=1}^{n} k^3$. This is also correct, because $\sum_{k=1}^{n} k^3 = \binom{n+1}{2}^2$. This is far from obvious; I may write a proof at some later point.

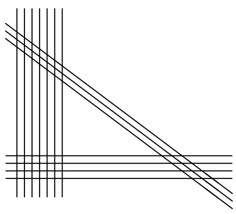
4. (a) Any two lines from the n give rise to a different intersection point, so there are $\binom{n}{2}$ points altogether.
(b) The x_i parallel lines will give no intersection points, resulting in these $\binom{x_i}{2}$ points 'being lost'. So now the number of intersection points is

$$\binom{n}{2} - \binom{x_1}{2} - \ldots - \binom{x_k}{2} = \frac{1}{2}(n^2 - n - x_1^2 + x_1 - \ldots - x_k^2 + x_k) = \frac{1}{2}(n^2 - x_1^2 - \ldots - x_k^2)$$

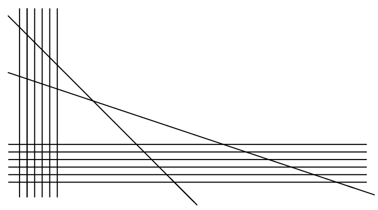
(the last equality being due to the fact that $n = x_1 + \ldots + x_k$).
(c) We need to find some $x_1, x_2, \ldots, x_k$ whose sum is 14 and whose squares sum to 74 (since $\frac{1}{2}(14^2 - x_1^2 - x_2^2 + \ldots - x_k^2) = 61$). Hence the possible values of the x_i's are

$$8, 2, 2, 1, 1 \text{ or } 7, 4, 3 \text{ or } 6, 6, 1, 1;$$

the last two cases are illustrated:



14 lines
61 intersections



5. Here are two different ways in which we can choose a colouring of the required type.

- We can start by choosing k items to be coloured. There are $\binom{n}{k}$ ways to do this.
- For each element of that set, we can choose one of the two colours. There are 2^k ways to do this.

This shows that there are $2^k\binom{n}{k}$ possibilities altogether.

Alternatively:

- We can start by choosing how many items to colour red. This is a number j with $0 \leq j \leq k$.
- Then we can choose j of the n items and colour them red. There are $\binom{n}{j}$ ways to do this.
- We now have $n-j$ uncoloured balls, and we choose $k-j$ of them and color them blue, to give k coloured balls altogether. There are $\binom{n-j}{k-j}$ ways to do this.

From this approach we see that the total number of possibilities is $\sum_{j=0}^{k}\binom{n}{j}\binom{n-j}{k-j}$.

In the homework, many students gave an algebraic proof, which is also valid, but not really in the spirit of this course. The argument is as follows: we have

$$\begin{aligned}\binom{n}{j}\binom{n-j}{k-j} &= \frac{n!}{j!(n-j)!}\frac{(n-j)!}{(k-j)!(n-k)!} = \frac{n!}{j!(k-j)!(n-k)!} \\ &= \frac{n!}{k!(n-k)!}\frac{k!}{j!(k-j)!} = \binom{n}{k}\binom{k}{j}.\end{aligned}$$

By taking the sum over j, we get

$$\sum_j \binom{n}{j}\binom{n-j}{k-j} = \binom{n}{k}\sum_j\binom{k}{j} = \binom{n}{k}2^k$$

as claimed.

The identity $\binom{n}{j}\binom{n-j}{k-j} = \binom{n}{k}\binom{k}{j}$ itself has a combinatorial interpretation, by counting the number of ways to colour j items red and $k-j$ items blue, in two different ways.

6. i) a) There are $\binom{2n}{n}$ ways to choose n people from $2n$.
On the other hand, this may be done by choosing n women from n and 0 men, or $n-1$ women and 1 man, or $n-2$ women and 2 men, and so on.
The number of ways to do this is

$$\binom{n}{n}\binom{n}{0} + \binom{n}{n-1}\binom{n}{1} + \binom{n}{n-2}\binom{n}{2} + \cdots + \binom{n}{0}\binom{n}{n}.$$

Using $\binom{n}{i} = \binom{n}{n-i}$, this gives

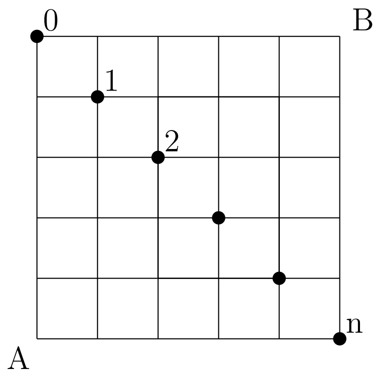
$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

b) $\binom{2n}{n}$ is the coefficient of x^n in the expansion of $(1+x)^{2n}$.

$$\begin{aligned}(1+x)^{2n} &= ((1+x)^n)^2 \\ &= \left(\sum_{k=0}^{n}\binom{n}{k}x^k\right)^2.\end{aligned}$$

So the coefficient of x^n is $\sum_{i=0}^{n}\binom{n}{i}\binom{n}{n-i}$.
Using $\binom{n}{i} = \binom{n}{n-i}$, this gives $\sum_{i=0}^{n}\binom{n}{i}^2 = \binom{2n}{n}$.

c) The number of shortest possible routes from bottom left point A to top right point B in a square $n \times n$ grid is $\binom{2n}{n}$.
Each such route passes through precisely one of the $n+1$ points, labelled $0, 1, 2, \ldots, n$ say, on the diagonal of the grid from top left to bottom right.



The no. of routes through point i

$$= (\text{no. of routes from } A \text{ to } i) \cdot (\text{no. of routes from } i \text{ to B}) .$$

Routes from A to i consist of $n - i$ units up and i units right and so the number of these is $\binom{n}{i}$.
Similarly, to get from i to B requires going i units up and $n - i$ units right, so again there are $\binom{n}{i}$ such routes.
So the total number of routes is $\sum_{i=0}^{n} \binom{n}{i}^2$.

ii) We may choose i women and i men, for $0 \leq i \leq n$.
There are $\binom{n}{i}\binom{n}{i}$ ways to choose i women and i men.
So the total is $\sum_{i=0}^{n} \binom{n}{i}^2$.
By the previous parts, this is $\binom{2n}{n}$.

iii) There must be at least one person in the subset to act as leader. For $1 \leq i \leq n$, there are $\binom{n}{i}$ ways to choose an i person subset and then i possibilities for leader of the subset.
So the total is $\sum_{i=1}^{n} i\binom{n}{i}$.
On the other hand, we may first pick a leader, in n ways, and then pick any subset of the remaining $n - 1$ people for them to lead, in 2^{n-1} ways.
[Alternative answer for iii):

$$\begin{aligned}
\sum_{i=1}^{n} i\binom{n}{i} &= \sum_{i=1}^{n} i\frac{n!}{i!(n-i)!} \\
&= \sum_{i=1}^{n} \frac{n!}{(i-1)!(n-i)!} \\
&= n\sum_{i=1}^{n} \frac{(n-1)!}{(i-1)!((n-1)-(i-1))!} \\
&= n\sum_{j=0}^{n-1} \frac{(n-1)!}{j!(n-1-j)!} \\
&= n(1+1)^{n-1} = n2^{n-1}. \ ]
\end{aligned}$$