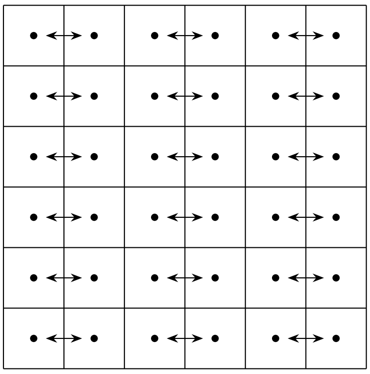
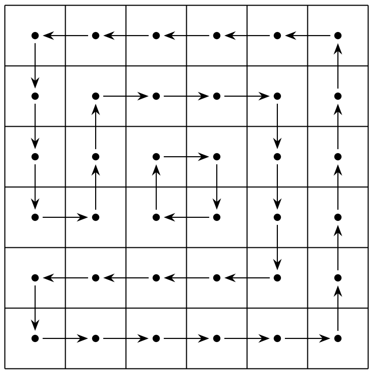
MAS334 COMBINATORICS 2017/2018

Solutions to Example Sheet 2 : Three basic principles

1. (a) In the cases where n is even it is easy to see that such moves are possible. For example, we can swap the piece in position $(i, 2j-1)$ with the one in position $(i, 2j)$, for $i = 1, \ldots, n$ and $j = 1, \ldots, n/2$. This is shown on the left below, for the case $n = 6$. However, there are also many other possibilities, such as the one shown on the right below.

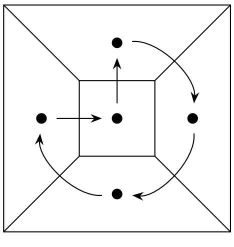



Comments: Note that a correct answer to this question **must** include an actual description of how to move the pieces. It is not enough to give some vague argument about why it should be possible.

(b) Now consider the case of an $n \times n$ board, where n is odd. Picture the board chequered with black and white squares in the usual chess-board fashion. As n is odd there will be more white squares than black, say (with $\frac{1}{2}(n^2+1)$ white and $\frac{1}{2}(n^2-1)$ black). Now each move to an adjacent square is from a black square to a white or *vice-versa*. But there are more white squares than black so not all the pieces moving from the white squares will fit onto the black ones. So such moves are impossible in the odd case.

Comments: Note that we need to consider the possibility of complicated patterns like the one shown on the right above, not just patterns where we swap adjacent pieces in pairs. Any proof in terms of dominos and swapping adjacent pairs is incomplete.

Note also the example of the following distorted board (which cannot be coloured black and white in the usual way):



The number of squares is five, which is odd, but it is still possible to move each piece to an adjacent square. This shows that any correct proof must use the colouring of squares, not just the fact that the number of squares is odd. Also, it is not enough to give a solution for the even case, and show that essentially the same solution does not work for the odd case. Just because one method fails for the odd case, we cannot conclude that every possible method fails.

2. Label pigeon-holes $1, 3, 5, \ldots, 2n-1$ and place each of the $n+1$ numbers into the pigeon-hole corresponding to its highest odd factor. Putting $n+1$ numbers into n pigeon-holes is bound to give two in the same pigeon-hole, by the Pigeon-Hole Principle. So two of the numbers will have the same highest odd factor, f say. So the numbers will be $2^r f$ and $2^s f$ for some integers r and s with $r < s$ say. But then clearly $2^s f = 2^{s-r} \times 2^r f$ and so one is a multiple of the other.

3. Put $a_m = 11\cdots1$ (with m digits), or equivalently $a_m = \sum_{i=0}^{m-1} 10^i = (10^m - 1)/9$. Fix $n > 1$, and put $\overline{a}_m = a_m \pmod{n}$, so $\overline{a}_m$ lies in the set $N = \{0, 1, \ldots, n-1\}$. As $|N| = n$ we see that the numbers $\overline{a}_1, \ldots, \overline{a}_{n+1}$ cannot all be different, by the pigeonhole principle. There must therefore exist indices p, q with $1 \leq p < q \leq n+1$ with $\overline{a}_q = \overline{a}_p$. This means that the number $b = a_q - a_p$ is divisible by n. Moreover, b has the form $1\cdots10\cdots0$, with $q-p$ ones followed by p zeros.

Comments: Here we are effectively placing the numbers $1, \ldots, n+1$ into pigeonholes $P_0, \ldots, P_{n-1}$ according to the values of $\overline{a}_i$. Some students said that every pigeonhole is nonempty, or that every pigeonhole contains more than one number. To see that this is not correct, consider the case $n = 1000$, for example. Then $\overline{a}_1 = 1$ and $\overline{a}_2 = 11$ and then $\overline{a}_3 = \overline{a}_4 = \overline{a}_5 = \cdots = 111$. This means that P_1 and P_{11} contain one number each, and everything else is in P_{111}, so all pigeonholes apart from P_1, P_{11} and P_{111} are empty.

4. Label 1000 pigeon-holes with the numbers $0, 1, \ldots, 999$. Put each of the numbers 7^0, 7^1, 7^2, ..., 7^{1000} into the pigeon-hole corresponding to its remainder on division by 1000. There are 1001 integers and 1000 pigeon-holes, so by the Pigeon-Hole Principle there are two numbers in the same pigeon-hole. These two powers of 7 have the same remainder on division by 1000, so they differ by a multiple of 1000.

5. We will use the notation $\lfloor x \rfloor$ to mean the integer part of x. So, for example, $\lfloor 1 \rfloor = \lfloor 1.5 \rfloor = \lfloor 1.999 \rfloor = 1$.

(a) Let B_i be the set of numbers in I divisible by p_i, so $N(i) = |B_i|$. These are the numbers $m\,p_i$ with $2 \leq m\,p_i \leq 10000$. As p_i is prime it is at least 2, so the inequality $2 \leq m\,p_i$ just means that $m \geq 1$. The inequality $m\,p_i \leq 10000$ means that $m \leq 10000/p_i$, and as m is an integer this is equivalent to $m \leq \lfloor 10000/p_i \rfloor$. Thus $N(i)$ is the number of possible values of m, which is just $\lfloor 10000/p_i \rfloor$. In particular, we have $p_1 = 2$ so $N(1) = \lfloor 10000/2 \rfloor = 5000$.

Similarly, $N(1, 2)$ is the size of the set $B_1 \cap B_2$, which is the set of elements of I that are divisible by $p_1 p_2 = 6$, and by the same logic we have $N(1, 2) = \lfloor 10000/6 \rfloor = 1666$. In general, we have

$$N(1, 2, \ldots, r) = \left\lfloor \frac{10000}{p_1 p_2 \cdots p_r} \right\rfloor .$$

(b) Hence the number of members of I with at least one of the properties 1,2 and 3 is

$$N(1) + N(2) + N(3) - N(1,2) - N(1,3) - N(2,3) + N(1,2,3)$$

which is $5000 + 3333 + 2000 - 1666 - 1000 - 666 + 333$ and equals 7334.

(c) Those 7334 numbers include 2, 3 and 5 themselves, but the other 7331 are multiples of these and so not prime. So there are at most 9999 - 7331 = 2668 primes less than ten thousand. (Note that we use 9999 here because 1 is not prime (by definition) and is not an element of I so $|I| = 9999$.)

Comments: If we consider $p_4 = 7$ as well as 2, 3 and 5 then the answer changes as follows: there are 7715 numbers in I that are divisible by 2, 3, 5 or 7; these are all composite apart from 2, 3, 5 and 7 themselves, so we have 7711 numbers that are known to be composite, leaving $9999 - 7711 = 2288$ numbers that might be prime. In fact only 1229 of these numbers are actually prime.

This question uses some of the same ideas as Euler's function (Theorem 35 in the notes) but the details are different. Some students tried to use Euler's formula directly, but it is not actually helpful here.

6. There are n^m functions since each of $f(1), f(2), \dots, f(m)$ has n choices. Now $N(1,2)$ (for example) is the number of functions f from $\{1, 2, \dots, m\}$ to $\{1, 2, \dots, n\}$ for which none of $f(1), f(2), \dots, f(n)$ equals 1 or 2; i.e. it is the number of functions from $\{1, 2, \dots, m\}$ to $\{3, 4, \dots, n\}$, which is $(n-2)^m$. Similarly $N(1,2,3) = (n-3)^m$ etc. Now, by inclusion/exclusion, the number of functions with none of the properties is

$$n^m - N(1) - \cdots - N(n) + N(1,2) + \cdots + N(n-1,n) - \cdots + (-1)^n N(1,2,\dots,n)$$

which gives the required answer

$$n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m - \binom{n}{3}(n-3)^m + \cdots$$

Having none of the properties is equivalent to **all** the numbers in $\{1, 2, \dots, n\}$ being used by f; i.e. we have counted the **surjections** from $\{1, 2, \dots, m\}$ to $\{1, 2, \dots, n\}$.

7. There is precisely one solution of the equation for each choice of position for $k-1$ dividers among the $n-1$ positions between n objects (excluding the end positions):

$$\underbrace{\bullet\bullet\bullet}_{x_1}\,|\,\underbrace{\bullet\bullet}_{x_2}\,|\,\bullet\bullet\cdots\bullet\,|\,\underbrace{\bullet\bullet}_{x_k}$$

Therefore there are $\binom{n-1}{k-1}$ solutions.

Let P_1 be the property that $x_1 > 5$, P_2 the property that $x_2 > 10$ and P_3 the property that $x_3 > 15$. We want the number of positive integer solutions with *none* of the properties P_1, P_2, P_3.

So this is

$$\begin{aligned}&\text{total no. of solutions} - \text{no. of solutions with at least one of the properties}\\&= \binom{19}{2} - \big(N(1) + N(2) + N(3) - N(1,2) - N(1,3) - N(2,3) + N(1,2,3)\big),\end{aligned}$$

where we have adopted usual Inclusion/Exclusion notation.

Now, if $x_1 > 5$, write $x_1 = 5 + x_1'$ where x_1' is a positive integer. So positive integer solutions of the original equation with $x_1 > 5$ correspond to positive integer solutions of $x_1' + x_2 + x_3 = 15$. So, by the first part, $N(1) = \binom{14}{2}$.

Similarly, $N(2) = \binom{9}{2}$, $N(3) = \binom{4}{2}$, $N(1,2) = \binom{4}{2}$, and $N(1,3) = N(2,3) = N(1,2,3) = 0$.

So the answer is

$$\binom{19}{2} - \binom{14}{2} - \binom{9}{2} - \binom{4}{2} + \binom{4}{2} = 171 - 91 - 36 = 44.$$