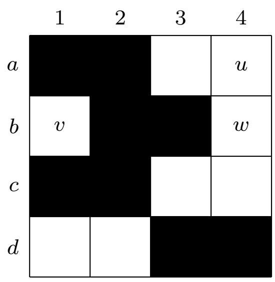
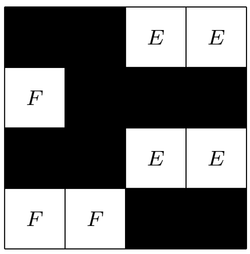
MAS334 COMBINATORICS

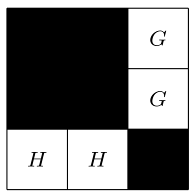
Solutions to Example Sheet 3 : Rook polynomials

1. This can be done using Theorems 43 and 46, or by just tabulating the solutions. If we use Theorem 43, it makes a big difference which square we start with. Some choices give a very efficient calculation but other choices require much more work, so we should choose carefully. Any choice of square will give us two new boards to consider. It will be good if at least one of those boards can be split into two disjoint pieces as in Theorem 46, and even better if both of the new boards can be split. Most people chose square u, v or w:

   $B =$ 

   Of these choices, w is the best, v is almost as good, and u is not so good. If we use w then we get $r_B(x) = r_C(x) + x\, r_D(x)$, where

   $C =$ 

   $D =$ 

   It is easy to find $r_E(x), \ldots, r_H(x)$ directly, and then we can use Theorem 46 to get $r_C(x)$ and $r_D(x)$, and Theorem 43 to get $r_B(x)$:

   $$r_E(x) = 1 + 4x + 2x^2 \qquad r_F(x) = 1 + 3x + x^2$$
   $$r_G(x) = 1 + 2x \qquad r_H(x) = 1 + 2x$$

   $$r_C(x) = r_E(x) r_F(x) = 1 + 7x + 15x^2 + 10x^3 + 2x^4$$
   $$r_D(x) = r_G(x) r_H(x) = 1 + 4x + 4x^2$$
   $$r_B(x) = r_C(x) + x\, r_D(x) = 1 + 8x + 19x^2 + 14x^3 + 2x^4.$$

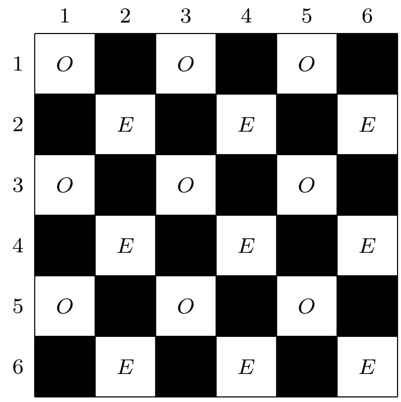
   We can also just tabulate the solutions. To avoid mistakes, I strongly recommend writing everything in dictionary order. We get the following table:

   - 1 rook: $a3$, $a4$, $b1$, $b4$, $c4$, $c4$, $d1$, $d2$ (8 possibilities)

- 2 rooks: $a3b1$, $a3b4$, $a3c4$, $a3d1$, $a3d2$, $a4b1$, $a4c3$, $a4d1$, $a4d2$, $b1c3$, $b1c4$, $b1d2$, $b4c3$, $b4d1$, $b4d2$, $c3d1$, $c3d2$, $c4d1$, $c4d2$ (19 possibilities)
- 3 rooks: $a3b1c4$, $a3b1d2$, $a3b4d1$, $a3b4d2$, $a3c4d1$, $a3c4d2$, $a4b1c3$, $a4b1d2$, $a4c3d1$, $a4c3d2$, $b1c3d2$, $b1c4d2$, $b4c3d1$, $b4c3d2$ (14 possibilities)
- 4 rooks: $a3b1c4d2$, $a4b1c3d2$ (2 possibilities).

This again gives $r_B(x) = 1 + 8x + 19x^2 + 14x^3 + 2x^4$.

2. The case $n = 6$ looks like this:



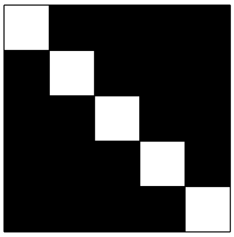
For half of the white squares, the row and column numbers are both odd. We have marked these squares O. For the other half of the white squares, the row and column numbers are both even. We have marked these squares E. Although we have only drawn the case $n = 6$, it works the same way whenever n is positive and even. Our board therefore splits as a disjoint union of two empty boards of size $(n/2) \times (n/2)$. There are $(n/2)!$ ways of placing a full set of non-challenging rooks on the O board, and the same number for the E board, so there are $(n/2)!^2$ ways of placing n non-challenging rooks on the white squares of the full board.

3. This was discussed in lectures. The single square has rook polynomial $1 + x$ and so the shaded board (in the $n \times n$ case) has rook polynomial $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n$. A layout of n non-challenging rooks on the unshaded board corresponds to a permutation with $1 \to 1$ etc avoided. Hence, using Theorem 51, the number of derangements of $\{1, 2, \ldots, n\}$ is

$$
\begin{aligned}
&n! - (n-1)!\binom{n}{1} + (n-2)!\binom{n}{2} - (n-3)!\binom{n}{3} + \ldots + (-1)^n 0!\binom{n}{n} \\
=&n! - n! + (n-2)!\frac{n!}{(n-2)!2!} - (n-3)!\frac{n!}{(n-3)!3!} + \ldots + (-1)^n \\
=&n!\left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots + (-1)^n\frac{1}{n!}\right).
\end{aligned}
$$

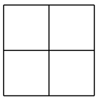
4. a) $1 + x$ is the rook polynomial of the full 1×1 board.

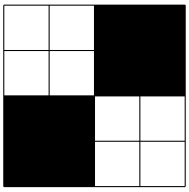
b) $(1+x)^n$ is the rook polynomial of an $n \times n$ board in which the diagonal squares are unshaded and all other squares are shaded. For example, $(1+x)^5$ is the rook polynomial of the following board:



This follows easily from Theorem 46, because we have a disjoint union of n copies of the full 1×1 board.

c,d) $1 + 4x + 2x^2$ is the rook polynomial of the full 2×2 board, as shown on the left below. Soo Theorem 46 tells us that $(1+4x+2x^2)^2$ is the rook polynomial of the disjoint union of two copies of the full 2×2 board, as shown on the right below.





e) This is not a rook polynomial: the coefficients in a rook polynomial are always non-negative integers, since they are the number of ways of placing non-challenging rooks. A coefficient -3 is not possible.

f) This is not a rook polynomial. Since the coefficient of x is 2, there would have to be two unshaded squares on the board. If the two squares share the same row or the same column, then there is no way of placing two non-challenging rooks, so the coefficient of x^2 would be zero. If the two squares do not share the same row or the same column, then there is precisely one way of placing two non-challenging rooks, so the coefficient of x^2 would be 1. But there is no way that the coefficient of x^2 can be 2.

5. This was discussed in lectures. Placing k non-challenging rooks on the m squares is like seating k people in a row of m seats with no two in adjacent seats. By Example 14, this can be done in $\binom{m+1-k}{k}$ ways: hence this is the coefficient of x^k in the rook polynomial, and the result follows.

6. This was also discussed in lectures. To find the rook polynomial of the shaded board use the standard method and choose the special square s as the bottom left-hand square. Then B_1 is a staircase of $2n-1$ squares and B_2 is a staircase of $2n-3$ squares. Hence (using the result from question 5) the rook polynomial of the shaded board is

$$\left(\dots + \binom{2n-k}{k} x^k + \dots\right) + x\left(\dots + \binom{2n-1-k}{k-1} x^{k-1} + \binom{2n-2-k}{k} x^k + \dots\right).$$

Hence for the shaded board the coefficient of x^k is $\binom{2n-k}{k} + \binom{2n-1-k}{k-1}$. Therefore, using Theorem 51, the number of ways of placing n rooks on the unshaded board is

$\sum_{k=0}^{n}(-1)^k(n-k)!(\binom{2n-k}{k}+\binom{2n-1-k}{k-1})$, and that is the number of ways of seating the wives.