MAS334 COMBINATORICS

Solutions to Example Sheet 4 : Applications of Hall's marriage theorem

1. We did not cover the Hungarian algorithm this year. I have included the solution in case you want to read about the Hungarian algorithm by yourself.

0	3	3	4	3
2	0	1	0	0
0	3	2	4	3
0	1	0	2	1
2	1	0	1	3

0s contained in 3 lines
Lowest not crossed is 1
Take 1 from uncrossed rows
Add 1 to crossed columns

0	2	3	3	2
3	0	2	0	0
0	2	2	3	2
0	0	0	1	0
2	0	0	0	2

0s contained in 4 lines
Lowest not crossed is 2
Take 2 from uncrossed rows
Add 2 to crossed columns

$\boxed{0}$	0	1	1	0
5	$\boxed{0}$	2	0	0
0	0	$\boxed{0}$	1	0
2	0	0	1	$\boxed{0}$
4	0	0	$\boxed{0}$	2

Jobs with five 0s;
e.g. Aa, Bb, Cc, De, Ed
all with a total of 4 in
the original table.

There are quite a few other choices of optimal allocation - any allocation with a total of 4 in the original table is correct.

2. Landau's Theorem says that a list $s_1, \ldots, s_8$ numbers can be the score list of an 8-person tournament iff

   (a) $s_1 + \ldots + s_8 = \binom{8}{2} = 28$; and

   (b) The sum of any k of the s_i's is at least $\binom{k}{2}$.

   If $s_1 \geq s_2 \geq \cdots \geq s_8$, then condition (b) is equivalent to the condition that the sum of the last k of the s_i must be at least $\binom{k}{2}$.

   For list (i), the sum of the entries is 29, so this cannot be the score list from a tournament.

   For list (ii), the sum of the last 4 entries is 5, which is less than $\binom{5}{2}$, so this cannot be the list of scores from a tournament. (A reminder of the reason: the last 4 players play $\binom{4}{2} = 6$ games against each other, so they must earn a total score of 6 from those games, even if they all lose against everyone else.)

   Now consider list (iii):

$$s_8 = 0 \geq 0 = \binom{1}{2} \qquad s_7 + s_8 = 1 \geq 1 = \binom{2}{2}$$

$$s_6 + s_7 + s_8 = 3 \geq 3 = \binom{3}{2} \qquad s_5 + s_6 + s_7 + s_8 = 6 \geq 6 = \binom{4}{2}$$

$$s_4 + \cdots + s_8 = 11 \geq 10 = \binom{5}{2} \qquad s_3 + \cdots + s_8 = 16 \geq 15 = \binom{6}{2}$$

$$s_2 + \cdots + s_8 = 22 \geq 21 = \binom{7}{2} \qquad s_1 + \cdots + s_8 = 28 = \binom{8}{2}.$$

   Landau's Theorem tells us that there exists a tournament with these scores. For the simplest example of such a tournament, suppose that the lower-numbered player wins every game except that player 4 beats player 1. Then the results and scores are as follows:

	1	2	3	4	5	6	7	8	
1		W	W	L	W	W	W	W	6
2	L		W	W	W	W	W	W	6
3	L	L		W	W	W	W	W	5
4	W	L	L		W	W	W	W	5
5	L	L	L	L		W	W	W	3
6	L	L	L	L	L		W	W	2
7	L	L	L	L	L	L		W	1
8	L	L	L	L	L	L	L		0

3. (a) Each player plays $n-1$ games, one against each of the other players. Each game results in a win for one of the players. So if player i wins w_i games, he/she loses $n-1-w_i$ games. So $l_i = n-1-w_i$.
   (b) Clearly there is a tournament in which the result of every game is the opposite to that in the given tournament. The scores of this tournament are $l_n, \ldots, l_1$.

   Alternatively, use Landau's Theorem:

   Since $w_1, \ldots, w_n$ are the scores of a tournament, by Landau's theorem, any r of them, say $w_{i_1}, \ldots, w_{i_r}$, add to at least $\binom{r}{2}$. Then

$$\begin{aligned}\sum_{j=1}^{r} l_{i_j} = \sum_{j=1}^{r} n-1-w_{i_j} &= r(n-1) - \sum_{j=1}^{r} w_{i_j} \\ &\leq r(n-1) - \binom{r}{2} \\ &= rn - r - \binom{r}{2} \\ &= rn - \binom{r+1}{2} \\ &= (n-1)+(n-2)+\cdots+(n-r).\end{aligned}$$

   Thus any r of the l_is add to at most $(n-1)+(n-2)+\cdots+(n-r)$ and, by Landau's Theorem, these are the scores of a tournament.

4. (a) Consider a tournament of 3 players (which we will call a *trio*). In total, $\binom{3}{2} = 3$ games are played. So there are three scores and the sum of the scores is 3. Each player plays two games, and so each individual score can be $0, 1$ or 2. If one player beats both of the others, we call them the *clear winner*. If there is a clear winner then we find that the scores are $2, 1, 0$. If there is no clear winner then we find that the scores must be $1, 1, 1$.

(b) Let W_i be the set of players who are beaten by player i, so $|W_i| = w_i$. Let C_i be the set of trios in which player i is the clear winner. To produce such a trio, we take player i together with a pair of players taken from W_i; so $|C_i| = \binom{w_i}{2}$.

(c) We need to find the number of trios with no clear winner. The total number of trios is $\binom{n}{3}$. The sets C_i are clearly disjoint, so the total number of trios that have a clear winner is just $\sum_i |C_i| = \sum_i \binom{w_i}{2}$. Thus, the total number of trios with no clear winner is $\binom{n}{3} - \sum_i \binom{w_i}{2}$.

5. (a) We are looking for tournaments in which the score list $s_1, \dots, s_n)$ is strictly decreasing. This means that we have n scores that are all different and all in the set $\{0, \dots, n-1\}$. It is clear that the only possibility is $s_1 = n-1$, $s_2 = n-2$, $\dots, s_{n-1} = 1$, $s_n = 0$, or more briefly $s_i = n - i$. We also claim that this pattern of scores can only occur if the lower-numbered player wins in every game. Indeed, P_1 has a score of $n-1$ and so must beat all of $P_2, \dots, P_n$. Player P_2 wins $n-2$ games but loses to P_1 and so must beat all of $P_3, \dots, P_n$. Player P_3 wins $n-3$ games but loses to P_1 and P_2 so must beat $P_4, \dots, P_n$, and so on.

   (b) Now imagine a tournament in which one score occurs precisely twice, and all the other scores are different. We will show that in fact this cannot happen. If it does happen, then there are precisely $n-1$ different scores altogether, and they all lie in the set $\{0, \dots, n-1\}$, so precisely one element of that set must be missing from the score list. Let i be the score that is missing, and let j be the score that is repeated. We have thus taken the score list from (a) but replaced i by j, so the total of the scores changes by $j - i$. However, the total of the scores must be $\binom{n}{2}$ in both cases, so $j - i = 0$, so $j = i$. But this is impossible, because i is missing from the score list and j is not. This contradiction shows that there can be no tournament of the type under consideration.

   (c) Now consider a tournament where $s_1 = x$ and $s_2 = s_3 = \cdots = s_n = y$ with $x > y$. By considering the sum of all the scores, we get $x + (n-1)y = \binom{n}{2} = n(n-1)/2$, which simplifies to $x = (n-1)(n/2 - y)$. By substituting this into the inequality $x > y$ and rearranging, we get $2y < n - 1$. On the other hand, Landau tells us that the sum of the last $n-1$ scores must be at least $\binom{n-1}{2} = (n-1)(n-2)/2$, so $(n-1)y \geq (n-1)(n-2)/2$, so $n - 2 \leq 2y$. We now have $n - 2 \leq 2y < n - 1$, with n and y being integers. This is only possible if $n = 2y + 2$ (so n must be even and $y = n/2 - 1$). Substituting this back into the relation $x = (n-1)(n/2 - y)$ gives $s_1 = x = n - 1$, so player P_1 must beat all the other players. For $k < n$ we see that the sum of the last k of the terms s_i is ky and $y = (n-2)/2 \geq (k-1)/2$ so $ky \geq k(k-1)/2 = \binom{k}{2}$. Thus, Landau's Theorem tells us that a tournament with these scores does indeed exist. In fact, in lectures we described how to use modular arithmetic to produce a tournament with an odd number of players and all scores the same. We just need to add one champion player to get the scores considered here.