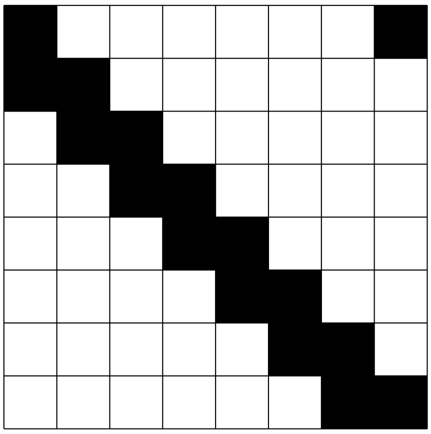
MAS334 COMBINATORICS

Solutions to Example Sheet 5 : Latin Squares and Designs

1. Translating to rooks, we need the number of ways of placing n non-challenging rooks on the unshaded part of an $n \times n$ board like this:



Rotating by 180^o, this gives the board of Example Sheet 3, Question 6. So, as in that question, the number is

$$\sum_{k=0}^{n}(-1)^k(n-k)!\left(\binom{2-k}{k}+\binom{2n-1-k}{k-1}\right).$$

2. Ignoring the x, we have the following multiplicities:

$$L(1)=4,\ L(2)=3,\ L(3)=2,\ L(4)=3,\ L(5)=1,\ L(6)=2,\ L(7)=0.$$

By Theorem 79, a $p \times q$ Latin rectangle extends to an $n \times n$ Latin square if and only if $L(i) \geq p+q-n$ for all i such that $1 \leq i \leq n$.

Taking $p=q=4$ and $n=7$ we need $L(i) \geq 4+4-7=1$ for each i such that $1 \leq i \leq 7$. Thus, the extension to a 7×7 Latin square is possible if and only if we take $x=7$ so as to change $L(7)$ to 1.

Taking $p=q=4$ and $n=6$, we need $L(i) \geq 4+4-6=2$ for each i such that $1 \leq i \leq 6$. Thus the extension to a 6×6 Latin square is possible if and only we take $x=5$ so as to change $L(5)$ to 2. To find an extension, we first add a new row. The possible values for the four columns are $\{2,6\}$, $\{3,5\}$, $\{4,5\}$ and $\{3,6\}$. Note that $L(3)=L(5)=L(6)=2$, which is the minimum allowed value, so 3, 5 and 6 are barely plausible for the extension problem. Thus, we need to ensure that 3, 5 and 6 appear in the new row. We can do this by taking $(6,5,4,3)$ as the new row. There is then a unique possible way to add a sixth row, namely $(2,3,5,6)$. This gives the following matrix:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 1 & 2 \\ 3 & 4 & 6 & 1 \\ 4 & 1 & 2 & 5 \\ 6 & 5 & 4 & 3 \\ 2 & 3 & 5 & 6 \end{bmatrix}$$

We now need to add another column. The possibilities for the six different rows are as follows:

$$a_1 \in \{5,6\}, a_2 \in \{3,4\}, a_3 \in \{2,5\}, a_4 \in \{3,6\}, a_5 \in \{1,2\}, a_6 \in \{1,4\}.$$

If we choose $a_1 = 5$, we find that a_3 must be 2, so a_5 must be 1, so a_6 must be 4, so a_2 must be 3, so a_4 must be 6. Thus, column 5 must be $(5,3,2,6,1,4)$. There is now only one possibility for column 6, namely $(6,4,5,3,2,1)$. The final result is the following Latin square:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 1 & 2 & 3 & 4 \\ 3 & 4 & 6 & 1 & 2 & 5 \\ 4 & 1 & 2 & 5 & 6 & 3 \\ 6 & 5 & 4 & 3 & 1 & 2 \\ 2 & 3 & 5 & 6 & 4 & 1 \end{bmatrix}$$

3. We are given that $p, q \leq n$ and that all the numbers $L(i)$ are the same, say $L(i) = k$. The total number of entries in the rectangle is pq, but each of the entries $1, \ldots, n$ occurs k times, so we must have $pq = nk$. We can rewrite this as $k = pq/n$, so for all i we have

$$\begin{aligned} E(i) &= L(i) + n - p - q = pq/n + n - p - q \\ &= (n^2 - pn - qn + pq)/n = (n-p)(n-q)/n. \end{aligned}$$

As $p, q \leq n$ we see that $n - p, n - q \geq 0$ and so $E(i) \geq 0$ for all i. It follows from Theorem 79 that L can be extended to an $n \times n$ Latin square.

4. There are 36 different possible correct answers for this question. Here is the simplest one:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$$

5. We let the blocks consist of all possible ways of choosing k varieties from the v varieties. So then we have $b = \binom{v}{k}$ blocks, each with k varieties.

   Then we need to consider a pair of varieties i, j. This pair occurs in all blocks where we have i, j and we can make any choice of the other $k-2$ varieties from the $v-2$ which are not i or j. So each pair occurs in $\binom{v-2}{k-2}$ blocks. Thus $\lambda = \binom{v-2}{k-2}$ and we do have a design.

   It then follows that each variety is in the same number of blocks r, with $r = bk/v = \binom{v}{k}k/v = \binom{v-1}{k-1}$. (Or, alternatively, each variety appears in $\binom{v-1}{k-1}$ blocks since there are $\binom{v-1}{k-1}$ ways to choose the other $k-1$ varieties in the block from the other $v-1$ varieties.)

6. **6a)** Since $23 = 4.5 + 3$, by Theorem 91 there is a $(23, 23, 11, 11, 5)$ design with starter block the quadratic residues of 23. Working mod 23, we have

$$\begin{aligned} &1^2 \equiv 1, \quad 2^2 \equiv 4, \quad 3^2 \equiv 9, \quad 4^2 \equiv 16, \quad 5^2 \equiv 2, \quad 6^2 \equiv 13, \\ &7^2 \equiv 3, \quad 8^2 \equiv 18, \quad 9^2 \equiv 12, \quad 10^2 \equiv 8, \quad 11^2 \equiv 6. \end{aligned}$$

So the quadratic residues are $\{1, 2, 3, 4, 6, 8, 9, 12, 13, 16, 18\}$. Take this as the first block and form each subsequent block by adding 1 and working mod 23.

b) The idea is to replace each block by its complement in the set of varieties. Certainly this will give b new blocks, each block containing $v - k$ varieties. To check this *is* a design we need to see that each pair appears in the same number of blocks.

We calculate how many new blocks the pair (i, j) appears in. This happens if and only if neither i nor j appeared in the corresponding old block. By the Inclusion/Exclusion Principle (Theorem 32), the number of times this happens is given by

$$\begin{array}{cccccc} & \text{no. of old blocks} & + & \text{no. of old blocks} & - & \text{no. of old blocks} \\ & \text{not containing } i & & \text{not containing } j & & \text{not containing } i \\ & & & & & \text{or not containing } j \\ = & (b-r) & + & (b-r) & - & (b-\lambda) \\ = & b - 2r + \lambda. & & & & \end{array}$$

Thus we do have a design. Clearly each variety appears in $b-r$ new blocks, so we have a $(v, b, b-r, v-k, b-2r+\lambda)$ design.

c) Suppose a $(v = 23, b = 23, r, k, \lambda)$ design exists. By Theorem 89, $r = \frac{bk}{v} = \frac{23k}{23} = k$. Also $r = \frac{\lambda(v-1)}{k-1}$, so $k(k-1) = 22\lambda$.

Recall that we are only interested in k in the range $1 < k < v = 23$. Now we need k such that $k(k-1)$ is divisible by $22 = 2.11$. Below we analyze the only possibilities for k satisfying these conditions and, in each case, we show that there is a corresponding design. The possibilities are as follows.

(a) $k = 11$, $k - 1 = 10$. Then $\lambda = 5$. This corresponds to a $(23, 23, 11, 11, 5)$ design and we constructed such a design in part a).

(b) $k - 1 = 11$, $k = 12$. Then $\lambda = 6$. This corresponds to a $(23, 23, 12, 12, 6)$ design. Such a design exists since we can use the complement of the first design, as in part b).

(c) $k = 22$, $k-1 = 21$. Then $\lambda = 21$. This corresponds to a $(23, 23, 22, 22, 21)$ design. Such a design can be constructed as follows. Put all the varieties except i in block i. Then we have 23 blocks with 22 varieties per block. Each pair (i, j) appears in all blocks except blocks i and j, so in 21 blocks.

7. **7a)** In any design, we have $\frac{bk}{v} = \frac{\lambda(v-1)}{k-1}$. So for such a design b must satisfy $\frac{3b}{11} = \frac{1.(11-1)}{(3-1)}$. That is, $3b = 55$. But this is impossible since b must be an integer.

   b) We have $r(k-1) = \lambda(v-1)$, with $k = 3$ and $\lambda = 1$. So $2r = v - 1$ and so v is odd. So v is $1, 3$ or 5 mod 6. On the other hand, $b(k-1)k = \lambda(v-1)v$ gives $6b = (v-1)v$. So 3 divides v or 3 divides $v - 1$. Hence v must be 1 or 3 mod 6.