MAS435: ALGEBRAIC TOPOLOGY
2017-18
EXERCISE SHEET 1

1. A *disconnection* of a topological space X is a pair of open and closed sets U, V partitioning X (i.e., $X = U \cup V$ and $U \cap V = \emptyset$). The disconnection is *trivial* if one of the two sets is empty. We say that x is *inseparable* from y if every open and closed set containing x also contains y, and in that case we write $x \sim y$.

(a) Show that $\sim$ is an equivalence relation.

(b) An equivalence class is called a *component* of X. Show that components are closed, but they need not be open.

(c) Write $\kappa(X) := X/\sim$. Calculate $\kappa(\mathbb{R})$, $\kappa(\mathbb{Q})$

(d) Let PI be the *Polish Interval*, a subset of $\mathbb{R}^2$ defined by

$$PI = \{(0, y) \mid -1 \leq y \leq 1\} \cup \{(x, sin(1/x)) \mid 0 < x < 1/\pi\}$$

Draw PI and calculate $\kappa(PI)$.

2. With the notation of Q1, show that if $f : X \longrightarrow Y$ is continuous then a disconnection of Y induces a disconnection of X. Deduce that f induces a map

$$f_* : \kappa(X) \longrightarrow \kappa(Y)$$

defined by $f_*([x]) = [f(x)]$, and that if $g : Y \longrightarrow Z$ is continous then $(gf)_* = g_* f_*$.

3. (a) Show that $\kappa([0, 1])$ has one point and conclude that there is a map

$$k : \pi_0(X) \longrightarrow \kappa(X)$$

defined by taking the path component containing x to the component containing x (i.e., $k([x]) = [x]$).

(b) Observe that k is surjective and show that it need not be injective.

4. Suppose X is a space so that every point has a path connected neighbourhood.

(a) Show that each path component of X is open and closed.

(b) Conclude that the partition into components coincides with the partition into path components and $k : \pi_0(X) \xrightarrow{\cong} \kappa(X)$ is a bijection.

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