# MAS435: ALGEBRAIC TOPOLOGY
# 2017-18
# WEEKLY EXERCISES (SEMESTER 2)

The weekly test and the weekly problem together contribute 20% to the assessment.

These Weekly Exercise problems are designed to be quick, and to provide a first opportunity to work with the ideas. The problems on the Exercise Sheets are designed to have more substance, and to develop ideas. They should add more depth to your understanding.

**Week 1** (Hand in at the lecture on Thursday of Week 2)

(a) Draw a picture of the octahedron (in $\mathbb{R}^3$). Write down a list of its triangular faces. Can you write down an analogue in $\mathbb{R}^n$ (with $2^n$ 'faces')?

(b) If $v_0, \dots, v_k$ are points in $\mathbb{R}^n$ show that $v_1 - v_0, v_2 - v_0, \dots, v_k - v_0$ is linearly independent if and only if $v_0 - v_1, v_2 - v_1, \dots, v_k - v_1$ is linearly independent.

(c) Write down all isomorphism classes of abelian groups of order 2, 4 and 8. Write down all isomorphism classes of abelian groups of order 6, 10, 15.

**Week 2** (Hand in at the lecture on Thursday of Week 3)

(a) If $Q$ consists of the four vertices of a unit square, find the Vietoris-Rips complex $VR_\epsilon(Q)$ for $\epsilon \geq 0$.

(b) Let $V$ be the set of 2-element subsets of $\{1, 2, 3, 4\}$. Define $G$ to be the 1-dimensional simplicial complex with edges consisting of elements of $V$ with an element in common. Draw $G$. Now let $Flag(G)$ be the flag complex of $G$ (i.e., $\sigma \in Flag(G)$ if and only if every edge of $\sigma$ is in $G$). Draw $Flag(G)$. Finally, let $K$ be the Cech nerve of the set of two element subsets of $\{1, 2, 3, 4\}$. Draw $K$. Is $K$ a flag complex?

(c) What is the minimum number of generators for $\mathbb{Z}/2 \times \mathbb{Z}/2$? What is the minimum number of generators for $\mathbb{Z}/2 \times \mathbb{Z}/3$? Is $\mathbb{Q}$ a finitely generated abelian group?

**Week 3** (Hand in at the lecture on Thursday of Week 4)

(a) If

$$0 \longrightarrow \mathbb{Z}/4 \longrightarrow A \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

and

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathbb{Z}/4 \longrightarrow B \longrightarrow \mathbb{Z}/4 \oplus \mathbb{Z}/2 \longrightarrow C \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

are exact, what are the possible isomorphism types for $A$, $B$ and $C$?

(b) Show that if

$$0 \longrightarrow A \longrightarrow B \longrightarrow \mathbb{Z} \longrightarrow 0$$

is exact then $B \cong A \oplus \mathbb{Z}$.

(c) If $K$ is an abstract simplicial complex and $\sigma \in K$ then the *link* of $\sigma$ is defined by

$$\mathrm{link}_\sigma(K) := \{\tau \in K \mid \tau \cap \sigma = \emptyset \text{ and } \tau \cup \sigma \in K\}.$$

Show that the link is a simplicial complex. Let $K$ be the octahedron, and draw a picture of the link of $\sigma$ when $\sigma$ is (i) a vertex (ii) an edge and (iii) a 2-simplex.

**Week 4** (Hand in at the lecture on Thursday of Week 5) Are the following sequences chain complexes? If so, calculate their homology? Are they exact?

$$0 \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{6} \mathbb{Z} \longrightarrow 0$$
$$0 \longrightarrow \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{3} \mathbb{Z}/6 \longrightarrow 0$$
$$\cdots \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \cdots$$
$$\cdots \longrightarrow \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \longrightarrow \cdots$$
$$0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \cdots$$
$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow \cdots$$

**Week 5** (Hand in at the lecture on Thursday of Week 6)

(a) Let $K$ be the pentagon (i.e., the simplicial complex with vertices $v_1, v_2, v_3, v_4, v_5$ and edges joining $v_i$ to $v_{i+1}$ (where the indices are considered mod 5)). Draw $K$. Guess $H_*(K)$ [You needn't tell me, but do write it down for yourself]. Calculate $H_*(K)$. Was your guess correct?

(b) Let $L$ be obtained from $K$ by adding the 2-simplex $\{v_1, v_3, v_5\}$ and its faces. Draw $L$. Guess $H_*(L)$ [You needn't tell me, but do write it down for yourself]. Calculate $H_*(L)$. Was your guess correct?

(c) Let $M$ be obtained by sticking together two hollow tetrahedra (the first with vertices $v_N, v_a, v_b, v_c$ and the second with vertices $v_a, v_b, v_c, v_S$). In other words $M$ has vertices $v_N, v_a, v_b, v_c, v_S$, and its 2-simplices are all sets of 3-vertices not containing both $v_N$ and $v_S$. Draw $M$. Calculate $H_2(M)$ (i.e., just the top homology).

**Week 6** (Hand in at the lecture on Thursday of Week 7)

Let $K$ be a simplicial complex, and $c_P K$ its $P$-cone. Now let $L = (c_P K)^{(d)}$ be the $d$-skeleton of $c_P K$ (i.e., it consists of all simplices of $c_P K$ of dimension $\leq d$). Draw pictures of this when $K = \Delta^2$ is the 2-simplex (for every relevant value of $d$).

Show that $H_i(L) = 0$ unless $i = 0$ or $i = d$. Observe that $H_d(L) \cong \mathbb{Z}^r$ for some $r$ ($r$ depends on $K$ and $d$ of course). Find $r$ in the cases for which you drew pictures above.

Optional: find $r = r(n, d)$ when $K = \Delta^{n-1}$ (the indexing is explained by the fact that $c_P \Delta^{n-1} \cong \Delta^n$).

**Week 7** (Hand in at the lecture on Thursday of Week 8)

For $h \geq 1$, the non-orientable surface $N(h)$ is obtained from a $2h$-gon by sticking together edges in pairs according to the surface word $z_1 z_1 z_2 z_2 \cdots z_h z_h$. Use the Mayer-Vietoris sequence to calculate $H_*(N(h))$. (You may assume all plausible assertions about triangulations without proof, and that homology is a homotopy invariant. )

What is a good way to distinguish orientable surfaces from non-orientable surfaces? Do you want to propose a definition of a connected orientable surface?

**Week 8** (Hand in at the lecture on Thursday of Week 9 (ie 24th April))

Recall that $\mathbb{R}P^n$ is $S^n/\sim$ where $P \sim -P$ identifies antipodal points. Let $B$ be the image in $\mathbb{R}P^n$ of the polar disc (of points $\leq 1/4$ from the north or south pole) and let $A$ be the image of the equatorial band (of points $\geq 1/4$ from both poles). Draw a picture for $n = 1$ and $n = 2$.

Note that $\mathbb{R}P^0$ is a point, and let $n \geq 1$. Observe that $B \simeq *$ and $A \cap B \cong S^{n-1}$. Prove $A \simeq \mathbb{R}P^{n-1}$.

Use the Mayer-Vietoris sequence to calculate $H_*(\mathbb{R}P^3)$ from the known value of $H_*(\mathbb{R}P^2)$.

[Note: Deducing $H_*(\mathbb{R}P^n)$ from $H_*(\mathbb{R}P^{n-1})$ is quite straightforward for $n$ odd, but when $n$ is even it is trickier. For complex projective space, both cases are easy.]

School of Mathematics and Statistics, Hicks Building, Sheffield S3 7RH. UK.
*E-mail address*: `j.greenlees@sheffield.ac.uk`