MAS435: ALGEBRAIC TOPOLOGY
2017-18
WEEKLY EXERCISES

The weekly test and the weekly problem together contribute 20% to the assessment.

These Weekly Exercise problems are designed to be quick, and to provide a first opportunity to work with the ideas. The problems on the Exercise Sheets are designed to have more substance, and to develop ideas. They should add more depth to your understanding.

Week 1 (Hand in at the lecture on Tuesday of Week 2)

The n-sphere S^n is defined by

$$S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_{n+1}^2 = 1\}.$$

(a) Draw pictures of S^n for $n = -1, 0, 1, 2$. Explain why these are really different.

(b) The north pole of S^n is the point $N = (0, \ldots, 0, 1)$. Write down a continuous bijection

$$s : \mathbb{R}^n \longrightarrow S^n.$$

[You might wish to use stereographic projection from the north pole.]

(c) Write down the 2-dimensional torus (i.e. the orientable surface of genus 1) as an explicit subset of $\mathbb{R}^n$ for some n of your choice. What do you think is the smallest possible n? Can you prove this?

Week 2 (Hand in at the lecture on Tuesday of Week 3)

(a) Suppose X is a topological space with topology $\mathcal{T}$ (i.e., $\mathcal{T}$ is the collection of open sets). Let $\mathcal{T}^*$ be the set of closed sets (i.e., the complements of elements of $\mathcal{T}$). We will call this the *cotopology* of X. Write down the axioms for a topology in terms of $\mathcal{T}^*$. Write down the condition for a map $f : X \longrightarrow Y$ to be continuous in terms of cotopologies.

(b) Write down all possible topologies on the a set X with two elements. Enthusiasts might wish to think about a set with three elements.

(c) Show that $\mathcal{T} = \{\emptyset, \mathbb{R}\} \cup \{(x, \infty) \mid x \in \mathbb{R}\}$ is a topology on $X = \mathbb{R}$.

Week 3 (Hand in at the lecture on Tuesday of Week 4)

(a) Draw sketches of $S^0 \times S^2$, $S^1 \times [0, 1]$, $S^1 \times S^1$, $S^2 \times [0, 1]$.

(b) Describe the topology of the quotient space $\mathbb{R}/[\mathbb{Q}]$ formed as the quotient of the real line in which all rational numbers have been identified to a point, with the quotient topology.

(c) Note that $\mathbb{Z}$ is a subgroup of $\mathbb{R}$, and consider the set $\mathbb{R}/\mathbb{Z}$ of left cosets with the quotient topology. Show that this is homeomorphic to the circle S^1.

Note that $\mathbb{Z}^2$ is a subgroup of $\mathbb{R}^2$, and consider the set $\mathbb{R}^2/\mathbb{Z}^2$ of left cosets with the quotient topology. Show that this is homeomorphic to the torus $S^1 \times S^1$.

Week 4 (Hand in at the lecture on Tuesday of Week 5)

(a) Show that $[0,1]/0 \sim 1$ (interval with ends identified) is homeomorphic to S^1.

(b) Give an example of a topological space X and a subset $Y \subset X$ where Y is compact but not closed.

(c) Show that the Polish point $PP = \{1/n \mid n \geq 1\} \cup \{0\}$ is not homeomorphic to the integers $\mathbb{Z}$.

Week 5 (Hand in at the lecture on Tuesday of Week 6)

(a) Let $S^1_r(x,y)$ be the circle in the plane centred at (x,y) and of radius r. Draw the Hawaiian Earrings $HE = \bigcup_{n\geq 1} S^1_{1/n}(0,1/n)$. Prove that HE is compact.

(b) Calculate π_0 of the following spaces. Give a careful justification for (i), (ii) and (iii). For the others just an answer and (optionally) a picture.

(i) $\mathbb{R} \setminus \{0\}$

(ii) S^2

(iii) $\mathbb{Q}$

(iv) $\mathbb{R} \setminus F$ where F has n points

(v) $\mathbb{R}^2 \setminus \{(0,0)\}$

(vi) $\mathbb{R}^2 \setminus \{(0,y) \mid y \in \mathbb{R}\}$

(vii) $S^1 \times S^1$

Week 6 (Hand in at the lecture on Tuesday of Week 8)

(a) Show that $\pi_0(X)$ is the set of homotopy classes of maps from the one point space $*$ to X by constructing a bijection

$$\pi_0(X) \cong [*, X].$$

(b) Thinking of $\mathbb{R}^2$ as the (x,y)-plane in $\mathbb{R}^3$ we have an inclusion $\mathbb{R}^2 \longrightarrow \mathbb{R}^3$. Show that the map $i : S^1 \longrightarrow S^2$ is homotopic to the constant map $c_N : S^1 \longrightarrow S^2$ at the north pole (ie $c_N(a) = N = (0,0,1)$ for all $a \in S^1$). Using the fact that S^2 is path connected, conclude that i is homotopic to c_E where the 'east pole' $E = (1,0,0)$.

(c) Show that $\mathbb{R}^3 \setminus S^2$ is homotopy equivalent to the subspace $\{0\} \cup \overline{S}^2$, where $\overline{S}^2$ is the sphere of radius 2. [Informally, we say that $\mathbb{R}^{n+1} \setminus S^n$ is homotopy equivalent to $S^n \coprod *$.]

Week 8 (Hand in at the lecture on Tuesday of Week 9)

(a) Show that the punctured 2-torus is homotopy equivalent to the figure eight space:

$$T^2 \setminus \{P\} \simeq S^1 \vee S^1$$

(You can take any model of T^2 that you find convenient and P to be any point of it, but you might find it convenient to take a square $[-1,1] \times [-1,1]$ with edges identified, and $P = (0,0)$. Similarly, you can take any model you like for the figure eight space (=the bouquet of two circles), but you might wish to take the image of the boundary of the above square in T^2. Draw a picture!)

(b) Show that if F consists of n real numbers, $\mathbb{R} \setminus F$ is homotopy equivalent to the discrete space with $n+1$ points. Hence give a new calculation of $\pi_0(\mathbb{R} \setminus F)$.

(c) (Optional, for interest but not for marks!) Consider the tangent bundle of S^n defined by

$$TS^n = \{(P, v) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid ||P|| = 1, P \cdot v = 0\}.$$

Show that $TS^n \simeq S^n$, with inverse homotopy equivalences $\pi : TS^n \longrightarrow S^n$ (defined by $\pi(P, v) = P$) and $\zeta : S^n \longrightarrow TS^n$ (defined by $\zeta(P) = (P, 0)$).

Week 9 (Hand in at the lecture on Tuesday of Week 10)

(a) Show that if X and Y are topological spaces with basepoints x_0 and y_0 then

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

[Hint: Use the map from left to right induced by $(\pi_X)_*$ in the first factor and $(\pi_Y)_*$ in the second factor, where $\pi_X : X \times Y \longrightarrow X$ and $\pi_Y : X \times Y \longrightarrow Y$ are the projections.]

(b) Suppose G is a topological group (we will just need to know that the multiplication $\mu : G \times G \longrightarrow G$ is continuous and that $\mu(g, e) = g = \mu(e, g)$ where e is the identity of G). If ω and σ are two loops in G based at e define a new loop $\omega * \sigma$ by

$$(\omega * \sigma)(s) = \mu(\omega(s), \sigma(s)).$$

Show that there are path homotopies

$$\omega \cdot \sigma \simeq \omega * \sigma \simeq \sigma \cdot \omega,$$

and hence deduce that $\pi_1(G, e)$ is abelian. [Exercise Sheet 2, Q2 gives examples.]

(c) (Optional) Exercise Sheet 2 for the effect of change of basepoint.

Week 10 (Hand in at the lecture on Tuesday of Week 11)

(a) Show that the map $f_n : S^1 \longrightarrow S^1$ defined by $f_n(z) = z^n$ is a covering map and show that the induced map $(f_n)_* : \pi_1(S^1, 1) \longrightarrow \pi_1(S^1, 1)$ is multiplication by n. [For the induced map, you may wish to use the statement of Theorem 7.4.]

(b) What is the fundamental group of the 2-torus? [You can write this down from things you know already!]

(c) Show that the quotient map $\pi : S^n \longrightarrow \mathbb{R}P^n$ is a covering map.

Week 11 (Hand in at the lecture on Tuesday of Week 12)

Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a monic polynomial of degree n with complex coefficients. We will show that it has a complex root (Fundamental Theorem of Algebra).

For $r \geq 0$, define $f'_r : S^1 \longrightarrow \mathbb{C}$ by $f'_r(e^{i\theta}) = p(re^{i\theta})$.

(a) Show that if r is sufficiently large, f'_r maps into $\mathbb{C} \setminus \{0\}$ and write $f_r : S^1 \longrightarrow \mathbb{C} \setminus \{0\}$ for the resulting map. Show that if r is sufficiently large, f_r is homotopic to the map g_n defined by $g_n(z) = z^n$.

(b) Suppose that $p(z)$ has no roots. Observe that f'_r maps into $\mathbb{C} \setminus \{0\}$ for all r and write $f_r : S^1 \longrightarrow \mathbb{C} \setminus \{0\}$ for the resulting map. Show that $f_r \simeq f_s$ for any $r \geq 0$.

(c) Show that if $n \geq 1$ then $g_n : S^1 \longrightarrow \mathbb{C} \setminus \{0\}$ is not homotopic to the constant map.

(d) Conclude from (a), (b) and (c) that if $n \geq 1$, the polynomial $p(z)$ has a root.

School of Mathematics and Statistics, Hicks Building, Sheffield S3 7RH. UK.
E-mail address: j.greenlees@sheffield.ac.uk