SUMMARY OF EXAMPLES FOR MAS435 (ALGEBRAIC TOPOLOGY)

N. P. STRICKLAND

1. Examples of spaces and simplicial complexes

Here $X \simeq Y$ means that X and Y are homeomorphic, and $X \cong Y$ means that X and Y are homotopy equivalent.

- $\mathbb{R}^n$ (contractible)
- Balls, cubes and simplices (all contractible)
- Skeleta of simplices
- Spheres, especially S^1, S^2 and S^3
- The Möbius strip $\cong S^1$
- Surfaces: the torus, the Klein bottle, the projective plane, the closed orientable surface $M(g)$ of genus g, the nonorientable surface $N(h)$.
- One-dimensional complexes, especially bouquets of circles, and letters of the alphabet
- Various complements: $S^n \setminus \text{point} \simeq \mathbb{R}^n$, $\mathbb{R}^n \setminus \text{point} \cong S^{n-1}$, $S^n \setminus S^m \simeq \mathbb{R}^n \setminus \mathbb{R}^m \cong S^{n-m-1}$, $S^2 \setminus (n+1\text{points}) \simeq \mathbb{R}^2 \setminus (n\text{points}) \cong \bigvee_{i=1}^n S^1$, $\mathbb{C} \setminus \mathbb{Z}$.
- The punctured torus $\cong S^1 \vee S^1$
- Spaces of matrices: $GL_n(\mathbb{R})$, $O(n)$ and so on. Especially $SO(2) \simeq U(1) \simeq S^1$ and $GL_2(\mathbb{R}) \simeq \{1, -1\} \times \mathbb{R}^3 \times S^1$.
- $\mathbb{R}P^n$ and $\mathbb{C}P^n$ for general n, either as quotients or as spaces of projection matrices. Especially $\mathbb{R}P^1 \simeq S^1$ and $\mathbb{R}P^2$.

2. Fundamental groups

- The following spaces are contractible and so have trivial fundamental group: $\mathbb{R}^n$, balls, cubes, simplices.
- The following spaces are not contractible but still have trivial fundamental group: S^n and $S^{n+k+1} \setminus S^k$ for any $n \geq 2$.
- The following spaces have $\pi_1(X) = \mathbb{Z}/2$: the projective spaces $\mathbb{R}P^n$ for $n \geq 2$
- The following spaces have $\pi_1(X) = \mathbb{Z}$: the circle S^1 and the Möbius strip.
- If X is a connected one-dimensional simplicial complex then X is always homotopy equivalent to $\bigvee_{i=1}^m S^1$ for some m, and $\pi_1(X)$ is a free group with m generators. This also applies if X is merely homotopy equivalent to a one-dimensional complex, as is the case for the punctured torus or $\mathbb{R}^2 \setminus (n\text{points})$. If $m = 0$ then X is contractible and so $\pi_1(X)$ is trivial. If $m = 1$ then $X \cong S^1$ and $\pi_1(X) = \mathbb{Z}$ (which is infinite and abelian). If $m > 1$ then $\pi_1(X)$ is a free group with more than one generator, so it is infinite and nonabelian. If we consider letters of the alphabet as one-dimensional complexes, then $\pi_1(B)$ is free on two generators, whereas A, D, O, P, Q, R have $\pi_1 = \mathbb{Z}$ and all other letters are contractible.
- The torus $T^2 = S^1 \times S^1$ has $\pi_1(T) = \mathbb{Z}^2$. More generally, the n-torus $T^n = (S^1)^n = S^1 \times \cdots \times S^1$ has $\pi_1(T^n) = \mathbb{Z}^n$. This is infinite and abelian.
- If X is a closed orientable surface of genus g, then we have the presentation

  $$\pi_1(X) = \langle a_1, \ldots, a_g, b_1, \ldots, b_g \mid [a_1, b_1][a_2, b_2] \cdots [a_g, b_g] = 1 \rangle.$$

  If $g = 0$ then $X \simeq S^2$ and $\pi_1(X) = 1$. If $g = 1$ then $X \simeq S^1 \times S^1$ and $\pi_1(X) = \mathbb{Z} \times \mathbb{Z}$. If $g > 1$ then $\pi_1(X)$ is infinite and not abelian. (It is also true, but not so easy to prove, that $\pi_1(X)$ is not isomorphic to a free group.)

- If X is the Klein bottle then $\pi_1(X)$ is an infinite nonabelian group that can be presented in various different ways, for example $\langle x, y \mid x^2 = y^2 \rangle$ or $\langle a, b \mid aba^{-1}b = 1 \rangle$.
- If $X = Y \times Z$ then $\pi_1(X) = \pi_1(Y) \times \pi_1(Z)$.
- If X is a disjoint union $Y \amalg Z$ then $\pi_1(X)$ is either $\pi_1(Y)$ (if the basepoint is in Y) or $\pi_1(Z)$ (if the basepoint is in Z).
- If $X = Y \vee Z$ then $\pi_1(X)$ is the free product of $\pi_1(Y)$ and $\pi_1(Z)$, obtained by combining the generators and relations for $\pi_1(Y)$ with those for $\pi_1(Z)$. In particular, if $\pi_1(Z) = 1$ then $\pi_1(X) = \pi_1(Y)$.
- If there is a covering map $p \colon X \to Y$ then the induced map $\pi_1(X) \to \pi_1(Y)$ is injective. Thus: if $\pi_1(Y)$ is trivial then $\pi_1(X)$ must be trivial, if $\pi_1(Y)$ is finite then $\pi_1(X)$ must be finite, and if $\pi_1(Y)$ is abelian then $\pi_1(X)$ must be abelian.
- Similarly, if X is a retract of Y then the map $\pi_1(X) \to \pi_1(Y)$ is again injective, so we have the same restrictions.

3. Homology

- The following spaces are contractible and so have $H_*(X) = \mathbb{Z}$: $\mathbb{R}^n$, balls, cubes, simplices.
- For $n > 0$ we have $H_0(S^n) = H_n(S^n) = \mathbb{Z}$, and $H_i(S^n) = 0$ for $i \neq 0, n$. We have mentioned several spaces that are homeomorphic or homotopy equivalent to S^n for some n, and so have the same homology: the Möbius strip, $\mathbb{R}P^1$, $SO(2)$ and $U(1)$ are all equivalent to S^1, and $GL_2(\mathbb{C}) \cong SU(2) \simeq S^3$, and $\mathbb{R}^n \setminus 0 \cong S^{n-1}$, and $S^{n+k+1} \setminus S^k \simeq \mathbb{R}^{n+k+1} \setminus \mathbb{R}^k \cong S^n$.
- For $X = \mathrm{skel}^n(\Delta^m)$ with $0 < n \leq m$ we have $H_0(X) = \mathbb{Z}$ and $H_n(X) = \mathbb{Z}^r$ where $r = \binom{m}{n+1}$. All other homology groups are zero. In fact X is homotopy equivalent to $\bigvee_{i=1}^r S^n$.
- For $X = \mathbb{C}P^n$ we have $H_0(X) = H_2(X) = \cdots = H_{2n}(X) = \mathbb{Z}$, and all other homology groups are trivial.
- The homology groups of $\mathbb{R}P^1 \simeq S^1$ are $(\mathbb{Z}, \mathbb{Z})$. The homology groups of $\mathbb{R}P^2$ are $(\mathbb{Z}, \mathbb{Z}/2)$.
- If X is a connected one-dimensional simplicial complex then X is always homotopy equivalent to $\bigvee_{i=1}^m S^1$ for some m, and $H_*(X) = (\mathbb{Z}, \mathbb{Z}^m)$.
- The torus $T^2 = S^1 \times S^1$ has $H_*(T^2) = (\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z})$. More generally, we have $H_k(T^n) = \mathbb{Z}^{\binom{n}{k}}$ for $0 \leq k \leq n$.
- If X is a closed orientable surface of genus g, then $H_*(X) = (\mathbb{Z}, \mathbb{Z}^{2g}, \mathbb{Z})$.
- For the Klein bottle we have $H_*(X) = (\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}/2)$.
- For any X, we have $H_0(X) = \mathbb{Z}\{\pi_0(X)\}$. Thus, if X has n path components, then $H_0(X) = \mathbb{Z}^n$. In particular, if X is path-connected then $H_0(X) = \mathbb{Z}$.
- If X is path connected then $H_1(X)$ is the abelianisation of $\pi_1(X)$.
- If X is a disjoint union $Y \amalg Z$ then $H_*(X) = H_*(Y) \oplus H_*(Z)$.
- If $X = Y \vee Z$ then $H_i(X) = H_i(Y) \oplus H_i(Z)$ when $i > 0$, but there is a small adjustment in degree zero. If Y has n path-components and Z has m path-components then $H_0(X) = \mathbb{Z}^{n+m-1}$ (whereas $H_0(Y) \oplus H_0(Z) = \mathbb{Z}^{n+m}$).