MAS435 Homework 1

Due: October 2, 2018

The n-sphere S^n is defined by

$$S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = 1\}$$

(a) Draw pictures of S^n for $n = -1, 0, 1, 2$. Explain why these are really different.

(b) The north pole of S^n is the point $N = (0, \ldots, 0, 1)$. Write down a continuous bijection

$$s : \mathbb{R}^n \to S^n \setminus \{N\}$$

[You might wish to use stereographic projection from the north pole.]

(c) Write down the 2-dimensional torus (i.e. the orientable surface of genus 1) as an explicit subset of $\mathbb{R}^n$ for some n. What do you think is the smallest possible n? Can you prove this?

MAS 435 Homework 2

Due: October 9, 2018

(a) Suppose X is a topological space with topology $\mathcal{T}$ (i.e. $\mathcal{T}$ is the collection of open sets). Let $\mathcal{T}^*$ be the set of closed sets (i.e. the complements of elements of $\mathcal{T}$). We will call this the <u>cotopology</u> of X. Write down the axioms for a topology in terms of $\mathcal{T}^*$. Write down the condition for a map $f: X \to Y$ to be continuous in terms of cotopologies.

(b) Write down all possible topologies on a set with three elements.

(c) Show that $\mathcal{T} = \{\emptyset, \mathbb{R}\} \cup \{(x, \infty) : x \in \mathbb{R}\}$ is a topology on $X = \mathbb{R}$.

MAS435 Homework 3

Due: October 16, 2018

(a) Draw sketches of $S^0 \times S^2$, $S^1 \times [0,1]$, $S^1 \times S^1$, $S^2 \times [0,1]$.

(b) Describe the open sets in the quotient space $\mathbb{R}/\mathbb{Q}$ where all rational numbers have been identified to a point.

(c) Note that $\mathbb{Z}^2$ is a subgroup of $\mathbb{R}^2$. Consider the set $\mathbb{R}^2/\mathbb{Z}^2$ of left cosets with the quotient topology. Show that this is homeomorphic to the torus $S^1 \times S^1$.

MAS435 Homework 4

Due: October 23, 2018

(a) Let $S^1_r(x,y)$ be the circle in the plane centered at (x,y) and of radius r. Draw the space

$$HE = \bigcup_{n \geq 1} S^1_{1/n}(0, \tfrac{1}{n})$$

and prove that HE is compact. (3 points.)

(b) Calculate π_0 of the following spaces. Give a careful justification for (i), (ii), and (iii). For the others, just give the answer and draw a picture. (7 points.)

(i) $\mathbb{R} \setminus \{0\}$

(ii) S^2

(iii) $\mathbb{Q}$

(iv) $\mathbb{R} \setminus F$ where F has n points

(v) $\mathbb{R}^2 \setminus \{(0,0)\}$

(vi) $\mathbb{R}^2 \setminus \{(0,y) : y \in \mathbb{R}\}$

(vii) $S^1 \times S^1$

MAS 435 Homework 5

Due: October 30, 2018

(a) Show that $\pi_0(X)$ is the set of homotopy classes of maps from the one point space to X by constructing a bijection

$$\pi_0(X) \cong [*, X]$$

(3 points)

(b) Thinking of $\mathbb{R}^2$ as the (x, y)-plane in $\mathbb{R}^3$, we have an inclusion $\mathbb{R}^2 \to \mathbb{R}^3$. Show that the resulting map $i : S^1 \to S^2$ is homotopic to the constant map $c_N : S^1 \to S^2$ at the north pole (i.e. $c_N(a) = N = (0,0,1)$ for all $a \in S^1$). Using the fact that S^2 is path connected, conclude that i is homotopic to c_E, the constant map at the "east pole" $E = (1,0,0)$.

(4 points)

(c) Show that $\mathbb{R}^3 \setminus S^2$ is homotopy equivalent to the subspace $\{0\} \cup \bar{S}^2$ where $\bar{S}^2$ is the sphere of radius 2. (3 points)

MAS 435 Homework 6

Due: November 13, 2018

(a) (5 points) Show that the punctured 2-torus is homotopy equivalent to the figure-8 space:

$$T^2 \setminus \{P\} \simeq S^1 \vee S^1$$

(You can take any model of T^2 that you find convenient and P to be any point in it, but you may find it convenient to take a square $[-1,1] \times [-1,1]$ with edges identified and $P = (0,0)$. Similarly, you can take any model you like for the figure-8 space (= the wedge of two circles), but you might wish to take the image of the boundary of the above square in T^2. Draw a picture!)

(b) (5 points) Show that if F consists of n real numbers, then $\mathbb{R} \setminus F$ is homotopy equivalent to the discrete space with $n+1$ points. Hence give a new calculation of $\pi_0(\mathbb{R} \setminus F)$.

(c) (Optional, not graded) Consider the tangent bundle of S^n defined by the formula

$$TS^n = \{(P, v) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} : \|P\| = 1,\ P \cdot v = 0\}.$$

Show that $TS^n \simeq S^n$ with inverse homotopy equivalences $\pi : TS^n \to S^n$ (defined by $\pi(P, v) = P$) and $\zeta : S^n \to TS^n$ (defined by $\zeta(P) = (P, 0)$).

MAS 435 Homework 7

Due: November 20, 2018

(a) Show that if X and Y are topological spaces with basepoints x_0 and y_0, then

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

[Hint: Use the map from left to right induced by $(\pi_X)_*$ in the first factor and $(\pi_Y)_*$ in the second factor where $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are the projections.] (5 points)

(b) Suppose G is a topological group. That is, G is a topological space with a continuous multiplication map $\mu : G \times G \to G$ giving G the structure of a group. (For this problem, we will just need to know that μ is continuous and that $\mu(g, e) = g = \mu(e, g)$ for all $g \in G$ where $e \in G$ is the identity element. If ω and σ are two loops in G based at e, define a new loop $\omega * \sigma$ by

$$(\omega * \sigma)(s) = \mu(\omega(s), \sigma(s)).$$

Show that there are homotopies

$$\omega \cdot \sigma \simeq \omega * \sigma \simeq \sigma \cdot \omega$$

relative to the basepoint. Deduce that $\pi_1(G, e)$ is abelian. (5 points)

MAS 435 Homework 8

Due: November 27, 2018

(a) Show that the map $f_n : S^1 \to S^1$ defined by $f_n(z) = z^n$ is a covering map and show that the induced map $(f_n)_* : \pi_1(S^1, 1) \to \pi_1(S^1, 1)$ is multiplication by n. (5 points)

(b) What is the fundamental group of the 2-torus? [You can write this down from things you know already!] (1 point)

(c) Define the <u>real projective space</u> $\mathbb{RP}^n$ to be the quotient of S^n by the equivalence relation that identifies a point $(x_1, \dots, x_{n+1}) \in S^n \subset \mathbb{R}^{n+1}$ with the point $(-x_1, \dots, -x_{n+1})$. Show that the quotient map $\pi : S^n \to \mathbb{RP}^n$ is a covering map. (4 points)

MAS 435 Homework 9
Due: December 4, 2018

Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a monic polynomial of degree n with complex coefficients. We will show that it has a complex root (Fundamental Theorem of Algebra).

For $r \geq 0$, define $f_r' : S^1 \to \mathbb{C}$ by

$$f_r'(e^{i\theta}) = p(re^{i\theta})$$

(a) Show that if r is sufficiently large, f_r' maps into $\mathbb{C} \setminus \{0\}$ and write $f_r : S^1 \to \mathbb{C} \setminus \{0\}$ for the resulting map. Show that if r is sufficiently large, f_r is homotopic to the map g_n defined by $g_n(z) = z^n$. (3 points)

(b) Suppose that $p(z)$ has no roots. Observe that f_r' maps into $\mathbb{C} \setminus \{0\}$ for all r and write $f_r : S^1 \to \mathbb{C}$ for the resulting map. Show that $f_r \simeq f_s$ for any $s \geq 0$. (3 points)

(c) Show that if $n \geq 1$, then $g_n : S^1 \to \mathbb{C} \setminus \{0\}$ is not homotopic to the constant map. (2 points)

(d) Conclude from (a), (b), and (c) that if $n \geq 1$, the polynomial $p(z)$ has a root. (2 points)