The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2014–2015**

Algebraic Topology - solutions **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (Standard)

(i) (a) Yes, f is homotopic to g. A homotopy α from g to f is given by

$$\alpha : I \times I \longrightarrow S^1$$

$$\alpha(x,t) = (cos(6\pi xt), sin(6\pi xt))$$

Note that $\alpha(x,0) = g(x)$ and $\alpha(x,1) = f(x)$ and α is continuous.

(2 marks)

(b) No, f is not loop homotopic to g. We know from the lecture that f represents 3 in $\pi_1(S^1) = \mathbb{Z}$ and g represents 0 in $\pi_1(S^1)$. If f was loop homotopic to g then they would represent the same element in $\pi_1(S^1)$. Since $0 \neq 3$ in $\mathbb{Z}$ f is not loop homotopic to g. ***(2 marks)***

(ii) We know from the lecture that if X is homotopy equivalent to Y then $\pi_1(X)$ is isomorphic to $\pi_1(Y)$. Since (again from the lecture) $\pi_1(\mathbb{R}P^2) = \mathbb{Z}/2\mathbb{Z}$ and $\pi_1(S^2) = 0$ they are not homotopy equivalent. ***(3 marks)***

(iii) (a) The unit disc D^2 in $\mathbb{R}^2$ is contractible. We use notation i for the inclusion of a point into D^2, i.e. $i : * \longrightarrow D^2$ where $i(*) = (0,0)$ and notation f for the map sending whole D^2 to the point, i.e. $f : D^2 \longrightarrow *$, where for all $(x,y) \in D^2$, $f((x,y)) = *$. We have $f \circ i = Id_*$. We need to show that there exists a homotopy α from $i \circ f$ to Id_{D^2}. Define α as follows

$$\alpha : D^2 \times I \longrightarrow D^2$$

$$\alpha((x,y),t) = (xt, yt)$$

Note that $\alpha((x,y),0) = 0 = i \circ f((x,y))$ and $\alpha((x,y),1) = (x,y) = Id_{D^2}((x,y))$ and since α is continuous that finishes the proof.

(3 marks)

(b) The complement of the disc in a plane $\mathbb{R}^2 \setminus D^2$ is not contractible. We know from the lecture that if X is contractible then $\pi_1(X)$ is trivial. We will show that $\mathbb{R}^2 \setminus D^2$ is homotopy equivalent to S^1. It will follow, that $\pi_1(\mathbb{R}^2 \setminus D^2) = \mathbb{Z} \neq 0$.

Firstly, we give two maps $f : \mathbb{R}^2 \setminus D^2 \longrightarrow S^1$ and $g : S^1 \longrightarrow \mathbb{R}^2 \setminus D^2$ as follows: $f((r,\theta)) = (1,\theta)$ and $g((1,\theta)) = (2,\theta)$ (using polar coordinates). Notice that since f is defined on $\mathbb{R}^2 \setminus D^2$ it is continuous. g is obviously continuous. Since the composite $f \circ g = Id_{S^1}$ we just need to define a homotopy α from $g \circ f$ to $Id_{\mathbb{R}^2 \setminus D^2}$. We do it as follows

$$\alpha : \mathbb{R}^2 \setminus D^2 \times I \longrightarrow \mathbb{R}^2 \setminus D^2$$

$$\alpha((r,\theta),t) = (2(1-t) + rt, \theta)$$

Note that $\alpha((r,\theta),1) = Id_{\mathbb{R}^2 \setminus D^2}((r,\theta))$ and $\alpha((r,\theta),0) = g \circ f((r\theta))$ and since α is continuous that finishes the proof. ***(3 marks)***

 Question 1 continued on next page

1 (continued)

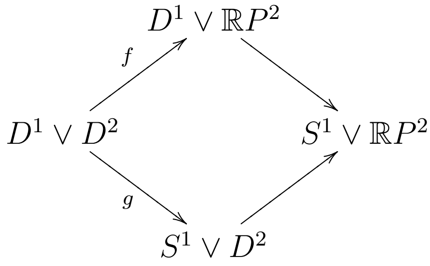
(iv) We know from the lecture that π_1 commutes with products and we know that $\pi_1(\mathbb{R}P^2) = \mathbb{Z}/2\mathbb{Z}$ and $\pi_1(T) = \pi_1(S^1 \times S^1) = \pi_1(S^1) \times \pi_1(S^1) = \mathbb{Z} \times \mathbb{Z}$. Therefore we have: $\pi_1(\mathbb{R}P^2 \times T) = \pi_1(\mathbb{R}P^2) \times \pi_1(S^1) \times \pi_1(S^1) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. ***(3 marks)***

(v) π_1 of a space depends on a basepoint. If a space X is a disjoint union of two path-connected components, for example $X = * \sqcup S^1$ then $\pi_1(X, *) = 0$, but $\pi_1(X, x) = \mathbb{Z}$ for any $x \in S^1$. However, we know from the lecture that $\pi_1(Y)$ does not depend on the choice of the basepoint if Y is path connected. ***(4 marks)***

2 (i) (a) (Standard) Since D^2 is contractible $\pi_1(S^1 \vee D^2) = \pi_1(S^1) = \mathbb{Z}$.

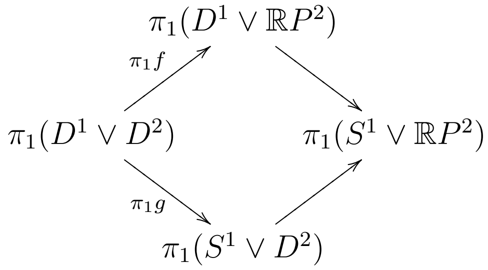
(1 mark)

(b) (We've done several examples of application of Van Kampen's theorem in the lectures.) To calculate $\pi_1(S^1 \vee \mathbb{R}P^2)$ by using Van Kampen's theorem we present our space as the following pushout:



(2 marks)

Here all discs are open discs, so that intersection of both spaces: $(D^1 \vee \mathbb{R}P^2) \cap (S^1 \vee D^2) = D^1 \vee D^2$ is an open neighbourhood of a joining point in $S^1 \vee \mathbb{R}P^2$. Note that all conditions of Van Kampen's theorem are satisfied, thus we can apply π_1 to this diagram



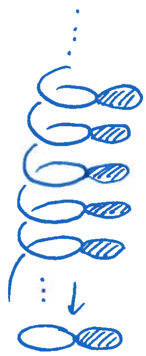
Since D^1 and D^2 are contractible we have $\pi_1(D^1 \vee D^2) = 0$, $\pi_1(D^1 \vee \mathbb{R}P^2) = \pi_1(\mathbb{R}P^2) = \mathbb{Z}/2\mathbb{Z}$ and $\pi_1(S^1 \vee D^2) = \pi_1(S^1) = \mathbb{Z}$. Notice that both maps on π_1 are trivial. Using Van Kampen's theorem we can conclude that $\pi_1(S^1 \vee \mathbb{R}P^2) \cong \mathbb{Z} * (\mathbb{Z}/2\mathbb{Z})$. ***(2 marks)***

2 (continued)

(c) (Standard knowledge from the lecture) The universal cover for S^1 is a real line $\mathbb{R}$ with the map $p : \mathbb{R} \longrightarrow S^1$ defined by $p(x) = (cos(2\pi x), sin(2\pi x))$. We know that from the lecture. ***(1 mark)***

The universal cover for D^2 is D^2 with the identity map. (we know that from the lecture) ***(1 mark)***

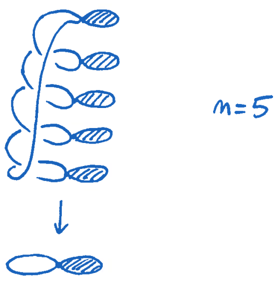
The universal cover for $S^1 \vee D^2$ is a real line with a copy of D^2 attached to every integer point. The covering map is defined as above on the real line and on every disc it is the identity map - we know it from the lecture.



(1 mark)

(d) (Similar to the example from the lecture, part of question from homework Week 11) Since $S^1 \vee D^2$ is path connected, locally path connected and semi locally simply connected we can use the classification theorem from the lecture. Connected covers of $S^1 \vee D^2$ (up to isomorphism) correspond to subgroups of $\pi(S^1 \vee D^2) \cong \mathbb{Z}$ by the classification theorem. We know that all subgroups of $\mathbb{Z}$ are of the form $n\mathbb{Z}$, for all $n \geqslant 0$. Trivial subgroup always corresponds to the universal cover, which is described above. When $n = 1$ the subgroup is actually the whole group, so it corresponds to the identity cover by $S^1 \vee D^2$. For $n > 1$ we have a copy of $S^1 = \mathbb{R}/n\mathbb{Z}$ with a copy of D^2 attached to every integer point. (So we have n copies of D^2). The covering map is n-fold cover of S^1 by S^1 and on every copy of D^2 it is an identity map, see picture:

2 (continued)

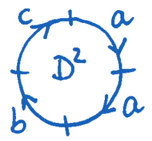


Since we mentioned all subgroups of $\mathbb{Z}$, the classification theorem proves that this is the full list of connected covering spaces (up to isomorphism).

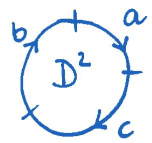
Question 2 continued on next page

2 (continued)

(ii) (As a question from homework Week 11, general statement with proof was given at the lecture) Construct a space with $\pi_1 = \langle a, b, c | a^2bc, acb \rangle$ and prove that your space's π_1 is as required.
We construct a space X as follows: first we take the wedge sum of 3 copies of S^1 (we name the generator of π_1 of each by a, b and c respectively): $\bigvee S^1$. Then we attach to it two discs D^2 via the maps on the boundary corresponding to the relations above, i.e. first disc will be attached via map $f : S^1 \longrightarrow \bigvee S^1$ shown on the picture:



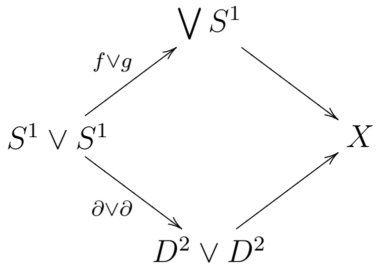
and the second disc will be attached via the map on the boundary of D^2 $g : S^1 \longrightarrow \bigvee S^1$ shown on the picture:



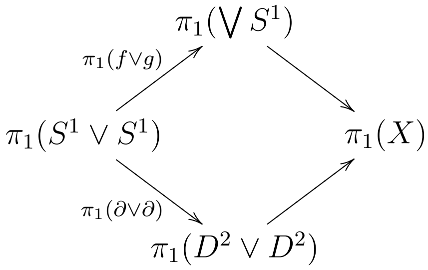
(3 marks)

We call this space X. Now we need to prove that $\pi_1(X)$ is as required. We will use the Van Kampen's theorem to do that. We can present X as a following pushout, where all discs are open and all copies of S^1 are made open by taking a homotopy equivalent spaces $(-\varepsilon, \varepsilon) \times S^1$:

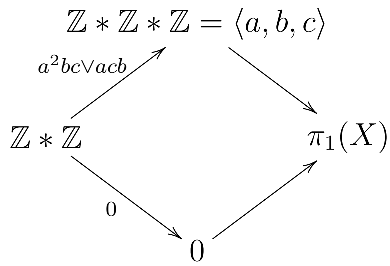
2 (continued)



where ∂ denotes the inclusion of the collar into D^2.
The above changes make it possible to use Van Kampen's theorem, but they won't change the corresponding homotopy groups, so we stick to the notation S^1:



We get the following diagram of groups and homomorphisms:



Using Van Kampen's theorem we get $\pi_1(X) = \langle a, b, c \mid a^2bc, acb \rangle$ as required.

(3 marks)

3 (i) (bookwork)

(a) A *singular n-simplex in X* is a continuous map $\Delta^n \to X$ where Δ^n is the standard n-simplex. In the singular chain complex of X, $C_n(X)$ is the free abelian group generated by all the singular n-simplices in X. The boundary map $\delta_n : C_n(X) \to C_{n-1}(X)$ is the group homomorphism determined on generators as follows: for an n-simplex $f : \Delta^n \to X$, $\delta_n(f) = \sum_{i=0}^{n}(-1)^i f_i$, where f_i is the restriction of f to the i-th face of Δ^n, $f_i : \Delta^{n-1} \cong \Delta_i^{n-1} \to X$.

(3 marks)

(b) $C_n(f) : C_n(X) \to C_n(Y)$ is the group homomorphism determined on generators by post-composition with f. That is, for $\sigma : \Delta^n \to X$ in $C_n(X)$, we define $C_n(f)(\sigma) = f\sigma : \Delta^n \to Y$ and we extend linearly.

(2 marks)

(c) Since all the maps are group homomorphisms, it's enough to check the required relation on generators. Let $\sigma \in C_n(X)$ be an n-simplex. Then

$$\begin{aligned} C_{n-1}(f)(\delta_n(\sigma)) &= C_{n-1}(f)\left(\sum_i (-1)^i \sigma|[v_0, \dots, \widehat{v_i}, \dots, v_n]\right) \\ &= \sum_i (-1)^i f\sigma|[v_0, \dots, \widehat{v_i}, \dots, v_n] \\ &= \sum_i (-1)^i C_n(f)(\sigma)|[v_0, \dots, \widehat{v_i}, \dots, v_n] = \delta_n C_n(f)(\sigma). \end{aligned}$$

(3 marks)

 Question 3 continued on next page

3 (continued)

(ii) (unseen, similar to homework problems)
Clearly, $H_{\geq 4} = 0$ and $H_3 = \ker \delta_3 = 0$. ***(1 mark)***

$$\begin{aligned}
\ker \delta_2 &= \{pb + qc + rd \,|\, p(15e - 6f) + q(30e - 12f) + r(15e - 6f) = 0, p, q, r \in \mathbb{Z}\} \\
&= \{pb + qc + rd \,|\, 15p + 30q + 15r = 0, -6p - 12q - 6r = 0, p, q, r \in \mathbb{Z}\} \\
&= \{pb + qc + rd \,|\, p + 2q + r = 0, p, q, r \in \mathbb{Z}\} \\
&= \{pb + qc + rd \,|\, r = -p - 2q, p, q \in \mathbb{Z}\} \\
&= \{pb + qc + (-p - 2q)d \,|\, p, q \in \mathbb{Z}\} \\
&= \{p(b - d) + q(c - 2d) \,|\, p, q \in \mathbb{Z}\} \\
&= \mathbb{Z}\{b - d\} \oplus \mathbb{Z}\{c - 2d\}
\end{aligned}$$

and

$$\operatorname{Im} \delta_3 = \mathbb{Z}\{7(b - d)\}.$$

So

$$H_2 = \frac{\ker \delta_2}{\operatorname{Im} \delta_3} = \frac{\mathbb{Z}\{b - d\} \oplus \mathbb{Z}\{c - 2d\}}{\mathbb{Z}\{7(b - d)\}} \cong \mathbb{Z}/7 \oplus \mathbb{Z}.$$

(3 marks)

$$\begin{aligned}
\ker \delta_1 &= \{pe + qf \,|\, 2pg + 5qg = 0, p, q \in \mathbb{Z}\} = \{pe + qf \,|\, 2p = -5q, p, q \in \mathbb{Z}\} \\
&= \{5re - 2rf \,|\, r \in \mathbb{Z}\} = \mathbb{Z}\{5e - 2f\}.
\end{aligned}$$

and

$$\operatorname{Im} \delta_2 = \mathbb{Z}\{15e - 6f\}.$$

So $H_1 = \mathbb{Z}/3$. ***(2 marks)***

$\operatorname{Im} \delta_1 = \mathbb{Z}\{g\}$, since $g = \delta_1(f - 2e)$, so $H_0 = 0$. ***(1 mark)***

(iii) (unseen, similar to homework problems)

(a) The chain complex is

$$0 \xrightarrow{0} \mathbb{Z}\{\alpha\} \xrightarrow{\delta_2} \mathbb{Z}\{a\} \oplus \mathbb{Z}\{b\} \xrightarrow{\delta_1} \mathbb{Z}\{x\} \oplus \mathbb{Z}\{y\} \xrightarrow{0} 0$$

where δ_2 is determined by $\delta_2(\alpha) = a$ and δ_1 is determined by $\delta_1(a) = 0$ and $\delta_1(b) = x - y$. ***(2 marks)***

Its homology groups are $H_{\geq 3} = 0, H_2 = \ker \delta_2 = 0, H_1 = \dfrac{\ker \delta_1}{\operatorname{Im} \delta_2} = \dfrac{\mathbb{Z}\{a\}}{\mathbb{Z}\{a\}} = 0$ and $H_0 = \dfrac{\mathbb{Z}\{x\} \oplus \mathbb{Z}\{y\}}{\mathbb{Z}\{x - y\}} \cong \mathbb{Z}$.

(2 marks)

(b) The space is a cone on a circle and hence contractible, so the homology groups are those of any contractible space. ***(1 mark)***

4 (i) (unseen, similar to homework problems)

(a) $C_*(X)$ is

$$0 \xrightarrow{0} \mathbb{Z}\{\alpha\} \oplus \mathbb{Z}\{\beta\} \xrightarrow{\delta_2} \mathbb{Z}\{a\} \oplus \mathbb{Z}\{b\} \oplus \mathbb{Z}\{c\} \xrightarrow{\delta_1} \mathbb{Z}\{x\} \oplus \mathbb{Z}\{y\} \oplus \mathbb{Z}\{z\} \xrightarrow{0} 0$$

where δ_2 is determined by $\delta_2(\alpha) = c$ and $\delta_2(\beta) = -c$ and δ_1 is determined by $\delta_1(a) = y - x$, $\delta_1(b) = z - x$ and $\delta_1(c) = 0$.

(2 marks)

$C_*(A)$ is

$$0 \xrightarrow{\quad 0 \quad} \mathbb{Z}\{c\} \xrightarrow{\delta_1 = 0} \mathbb{Z}\{x\} \xrightarrow{\quad 0 \quad} 0$$

(1 mark)

$C_*(X, A)$ is

$$0 \xrightarrow{0} \mathbb{Z}\{\alpha\} \oplus \mathbb{Z}\{\beta\} \xrightarrow{\delta_2} \mathbb{Z}\{a\} \oplus \mathbb{Z}\{b\} \xrightarrow{\delta_1} \mathbb{Z}\{y\} \oplus \mathbb{Z}\{z\} \xrightarrow{0} 0$$

with maps induced from those in $C_*(X)$, so $\delta_2 = 0$, $\delta_1(a) = y$, $\delta_1(b) = z$. ***(1 mark)***

(b) $H_{\geq 3}(X, A) = 0$, $H_2(X, A) = \ker \delta_2 = \mathbb{Z}\{\alpha\} \oplus \mathbb{Z}\{\beta\}$. ***(1 mark)***

$$H_1(X, A) = \frac{\ker \delta_1}{\operatorname{Im} \delta_2} = 0.$$ ***(1 mark)***

$$H_0(X, A) = \frac{\mathbb{Z}\{y\} \oplus \mathbb{Z}\{z\}}{\operatorname{Im} \delta_1} = \frac{\mathbb{Z}\{y\} \oplus \mathbb{Z}\{z\}}{\mathbb{Z}\{y\} \oplus \mathbb{Z}\{z\}} = 0.$$ ***(1 mark)***

(c) The space X consists of two cones on a circle glued along the circle. The subspace A is this circle. The quotient space has the homotopy type of a wedge of two 2-spheres. The reduced homology groups of this space are the ones calculated in part (b), since we know the reduced homology of a 2-sphere is a single copy of $\mathbb{Z}$ in degree 2 and that reduced homology takes wedges to direct sums. ***(3 marks)***

 Question 4 continued on next page

4 (continued)

(ii) (unseen)

(a) If $A \cap B \neq \emptyset$, we have the reduced homology version of the Mayer-Vietoris long exact sequence:

$$\cdots \longrightarrow \widetilde{H}_n(A \cap B) \to \widetilde{H}_n(A) \oplus \widetilde{H}_n(B) \longrightarrow \widetilde{H}_n(X) \to \widetilde{H}_{n-1}(A \cap B) \longrightarrow \widetilde{H}_{n-1}(A) \oplus \widetilde{H}_{n-1}(B) \longrightarrow \cdots$$

$$\cdots \longrightarrow \widetilde{H}_0(A) \oplus \widetilde{H}_0(B) \longrightarrow \widetilde{H}_0(X) \longrightarrow 0$$

(2 marks)

By assumption, $\widetilde{H}_n(X)$ appears between zero groups in an exact sequence, for all $n \geq 0$, so

$$\widetilde{H}_n(X) = \ker(\text{outgoing map}) = \operatorname{Im}(\text{incoming map}) = 0.$$

(2 marks)

(b) Write $Y = X \cup C$, where $X = A \cup B$. By part (a), $\widetilde{H}_n(X) = 0$ for all $n \geq 0$. ***(1 mark)***

Now consider $X \cap C = P \cup Q$ where $P = A \cap C$ and $Q = B \cap C$. ***(1 mark)***

We can apply the unreduced M-V sequence to $P \cup Q$, using $P \cap Q = A \cap B \cap C$ and this time, by the argument seen in part (a), we will get for $n \geq 1$, $\widetilde{H}_n(P \cup Q) = H_n(P \cup Q) = 0$. ***(1 mark)***

Now we apply M-V one more time for $Y = X \cup C$. We will have $\widetilde{H}_n(Y) = H_n(Y)$ appearing between zero groups, this time for $n \geq 2$, so $\widetilde{H}_n(Y) = 0$ for $n \geq 2$ as required. ***(1 mark)***

Finally, for the example, we can take $Y = S^1$, covered by three nicely overlapping open arcs A, B, C such that A, B, C and all the pairwise intersections are contractible, but the triple intersection is empty. Here $\widetilde{H}_1(S^1) = \mathbb{Z}$. ***(2 marks)***

End of Question Paper