The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2017–2018**

MAS435 Algebraic Topology **2 hours 30 minutes**

Answer ***four*** *questions. You are advised* ***not*** *to answer more than four questions: if you do, only your best four will be counted.*

1 (i) (a) What is a *topological space*? If X and Y are topological spaces define the *product topology* on $X \times Y$. ***(5 marks)***

(b) Suppose that T, X and Y are topological spaces and $f : T \longrightarrow X \times Y$. Writing $\pi_X : X \times Y \longrightarrow X$ and $\pi_Y : X \times Y \longrightarrow Y$ for the projections, show that if the composites $\pi_X \circ f$ and $\pi_Y \circ f$ are continuous, so too is f. ***(4 marks)***

(ii) (a) If X is a topological space and $a, b \in X$ what is a *path* from a to b? ***(2 marks)***

(b) Suppose that ω is a path from a to b and σ is a path from b to c define the *concatenated* path $\omega \cdot \sigma$.

Consider the relation $a \sim b$ if there is a path from a to b. Show that this is an equivalence relation. (We let $\pi_0(X)$ denote the set of equivalence classes). ***(5 marks)***

(c) If $f : X \longrightarrow Y$ is a continuous function, explain how to define the induced map $f_* : \pi_0(X) \longrightarrow \pi_0(Y)$. ***(3 marks)***

(d) Show that the map $f : \pi_0(X \times Y) \longrightarrow \pi_0(X) \times \pi_0(Y)$ induced by the projections is a bijection. ***(6 marks)***

2 (i) (a) What is a *covering map*? ***(4 marks)***

(b) State the Path Lifting Lemma for a covering map $p : Y \longrightarrow X$, and explain how it can be used to define a function

$$\ell : \pi_1(X, x_0) \longrightarrow p^{-1}(x_0),$$

where $x_0 \in X$ (you need not check that your definition is independent of the choices you make). State conditions under which ℓ is a bijection. ***(8 marks)***

(ii) (a) Consider the torus $T = S^1 \times S^1$ and the self-map $f : T \longrightarrow T$ defined by $f(w, z) = (-w, \overline{z})$ (where w, z are complex numbers of modulus 1) and notice that f^2 is the identity. Take the quotient space $K = T/\sim$ where $(w, z) \sim f(w, z)$ and give it the quotient topology. Show that the quotient map $p : T \longrightarrow K$ is a covering map. ***(4 marks)***

(b) Choose basepoints $\tilde{x}_0 = (1, 1)$ and $x_0 = p(\tilde{x}_0)$. Show that $p_* : \pi_1(T, \tilde{x}_0) \longrightarrow \pi_1(K, x_0)$ is injective. Let $\tilde{\sigma}$ be any path from $f(\tilde{x}_0)$ to $\tilde{x}_0$ and let $h \in \pi_1(K, x_0)$ be the class of the loop $\sigma := p \circ \tilde{\sigma}$. Show that for any element $g \in \pi_1(K, x_0)$, either g or gh is in the image of p_*. Deduce that $\pi_1(K, x_0)$ is generated by three elements. ***(9 marks)***

3 (i) What is a *chain complex* of abelian groups? What is the *homology* of such a chain complex? ***(5 marks)***

(ii) Show that if K is a n-dimensional simplicial complex, then $H_n(K)$ is a free abelian group. Show that if L is a subcomplex of K which includes all simplices of dimension $\leq d$ then $H_i(L) = H_i(K)$ for $i \leq d-1$. ***(7 marks)***

(iii) Let Δ^n be the standard n-simplex with vertex set $\{e_0, e_1, \ldots, e_n\}$. Write down $H_*(\Delta^n)$.

Let $(\Delta^n)^{(d)}$ be the simplicial complex of all faces of dimension $\leq d$. Draw pictures of $(\Delta^3)^{(k)}$ for $k = 0, 1, 2$. ***(4 marks)***

Show that $H_i((\Delta^n)^{(d)}) = 0$ unless $i = 0$ or $i = d$. For $n \geq 3$, calculate the homology of $(\Delta^n)^{(n-1)}$ and $(\Delta^n)^{(n-2)}$. ***(9 marks)***

4 (i) State the Mayer-Vietoris Theorem for calculating the homology of a simplicial complex $K = L \cup M$ expressed as the union of two subcomplexes L and M. ***(5 marks)***

(ii) Let X be formed by sticking a Möbius strip to a 2-torus T^2 by identifying the boundary circle with some circle in T^2. Suppose X may be triangulated using a simplicial complex $K = L \cup M$ with L being a triangulation of the 2-torus T^2, and let M being a triangulation of the Möbius strip.

Write down $H_*(L), H_*(M), H_*(L \cap M)$ and identify the map induced by the inclusion $L \cap M \longrightarrow M$, making any assumptions about the triangulations that are convenient. ***(8 marks)***

Write down the Mayer-Vietoris long exact sequence for $K = L \cup M$, and identify $H_0(K), H_2(K)$. Identify two possibilities for $H_1(K)$ and show that they both occur. ***(12 marks)***

5 Are the following true or false. Justify your answers.

(i) Any continuous self-map of the closed unit ball $\overline{B}^3$ in $\mathbb{R}^3$ has a fixed point. ***(5 marks)***

(ii) Writing $d(P, Q)$ for the Euclidean distance from P to Q, the space $X := \{(x, y) \in \mathbb{R}^2 \mid d((x, y), (n, 0)) < 1/2 \text{ for some } n \in \mathbb{Z}\}$ is homeomorphic to $Y := \{(x, y) \in \mathbb{R}^2 \mid (x, y) \notin \mathbb{Z} \times \{0\}\}$. ***(5 marks)***

(iii) There is a covering map $K^2 \longrightarrow T^2$ from the Klein bottle to the torus. ***(5 marks)***

(iv) One may remove a finite number of points from $\mathbb{R}^2$ and obtain a space homotopy equivalent to the projective plane $\mathbb{R}P^2$. ***(5 marks)***

(v) The space X obtained by deleting the z axis from $\mathbb{R}^3$ is homotopy equivalent to a 1-dimensional complex. ***(5 marks)***

End of Question Paper