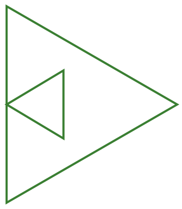
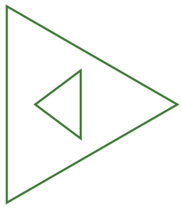
Algebraic Topology

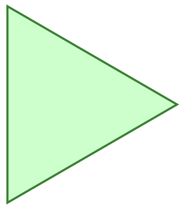
(1) Consider the following spaces:



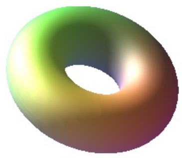
X_0



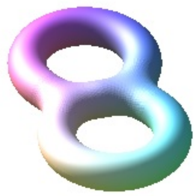
X_1



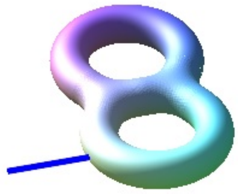
X_2



X_3



X_4



X_5

$$X_6 = (S^1 \times S^1) \setminus \{(1,1)\} \qquad X_7 = GL_2(\mathbb{R}) = \{A \in M_2(\mathbb{R}) \mid \det(A) \neq 0\}$$
$$X_8 = \mathbb{R} \qquad X_9 = \{(u,v) \in \mathbb{C}^2 \mid 1 \leq |u| \leq 2 \leq |v| \leq 3\}.$$

(Here X_3 and X_4 are closed orientable surfaces, and X_5 is the union of X_4 with a line segment with one endpoint lying on X_4. Everything else should be clear.)

(a) These 10 spaces can be grouped into 5 pairs $\{X_i, X_j\}$ such that X_i is homotopy equivalent to X_j. Find these pairs, and justify your answers. In each case you should prove that X_i is homotopy equivalent to X_j, and also that it is not homotopy equivalent to any of the other spaces. **(25 marks)**

(b) For each pair $\{X_i, X_j\}$ as in (a), prove that X_i is not homeomorphic to X_j. (In one case you may need to appeal to some geometric intuition, but you should be able to give a more formal proof in the other four cases.) **(15 marks)**

Solution:

(a) **This will need to be marked as a whole. There will be** 5 **marks for correct identification of the pairs,** 10 **marks for justifying why they are homotopy equivalent, and a further** 10 **marks for explaining why there are no further equivalences.** [15]

1. X_0 consists of two circles meeting at a single point and so is homeomorphic to the figure eight. This is in turn homotopy equivalent to the punctured torus X_6, as explained in Example 15.26 and the associated interactive demonstration.
2. X_1 is homeomorphic to the union of two disjoint circles. On the other hand, Example 4.9 shows that the space $X_7 = GL_2(\mathbb{R})$ is homeomorphic to $\mathbb{R}^3 \times S^1 \times \{1, -1\}$, so it is homotopy equivalent to $S^1 \times \{1, -1\}$, which is again a union of two disjoint circles. Thus, X_1 is homotopy equivalent to X_7.
3. X_2 and X_8 are both contractible and so are homotopy equivalent to each other.

4. X_3 is just the torus $S^1 \times S^1$. There is a homeomorphism

$$p\colon [0,1]^2 \times X_3 = [0,1]^2 \times S^1 \times S^1 \to X_9$$

given by $p(s,t,u,v) = ((1+s)u,\ (2+t)v)$, and $[0,1]^2$ is contractible, so X_3 is homotopy equivalent to X_9.

5. The spaces X_4 and X_5 are homotopy equivalent. Indeed, the extra interval in X_5 can be parametrised as $\{u(t) \mid 0 \leq t \leq 1\}$, with $u(0)$ being the end lying in X_4. We have an evident inclusion $i\colon X_4 \to X_5$ and a retraction $r\colon X_5 \to X_4$ given by $r(u(t)) = u(0)$ and $r(x) = x$ for all $x \in X_4$. Then $r \circ i$ is equal to the identity. We can also define $h\colon [0,1] \times X_5 \to X_5$ by $h(s, u(t)) = u(st)$ and $h(s,x) = x$ for all $x \in X_4$. This gives a homotopy $i \circ r \simeq \mathrm{id}$, so we have a homotopy equivalence as claimed.

If two spaces are homotopy equivalent, then they have isomorphic homology. We can tabulate the homology groups of the X_i as follows:

	H_0	H_1	H_2
X_0, X_6	$\mathbb{Z}$	$\mathbb{Z}^2$	0
X_1, X_7	$\mathbb{Z}^2$	$\mathbb{Z}^2$	0
X_2, X_8	$\mathbb{Z}$	0	0
X_3, X_9	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}$
X_4, X_5	$\mathbb{Z}$	$\mathbb{Z}^4$	$\mathbb{Z}$

As all the lines are different, there are no additional homotopy equivalences [10]. **There are also valid approaches using π_0 and π_1. They are less clear and efficient, but can also be given full marks if done correctly.**

(b) The space X_0 is compact but X_6 is not, so X_0 is not homeomorphic to X_6 [3]. Similarly X_1 is compact but X_7 is not [3], and X_2 is compact but X_8 is not [3]. Next, X_5 can be disconnected by removing a single point, but X_4 cannot, so X_4 and X_5 are not homeomorphic [3]. Finally X_3 and X_9 are not homeomorphic because X_3 is 2-dimensional and X_9 is 4-dimensional [3]. (This is not quite a complete proof, because we have not given a formal definition of dimensionality. The Invariance of Domain Theorem does most of what we need, but a bit more discussion would be required.)

(2)

(a) Let A and B be finite abelian groups such that $|A|$ and $|B|$ are coprime.

(i) What can you say about homomorphisms from A to B? **(10 marks)**

(ii) Now suppose we have a short exact sequence $A \to U \to B$ of abelian groups. By considering the classification of finite abelian groups, or otherwise, what can you say about U? **(15 marks)**

(b) Let X be a topological space, with open subspaces U and V such that $X = U \cup V$. Suppose that U, V, X and $U \cap V$ are all path-connected, and that for all $k > 0$ we have $H_k(U \cap V) = \mathbb{Z}/2^k$ and $H_k(U) = \mathbb{Z}/3^k$ and $H_k(V) = \mathbb{Z}/5^k$. Calculate $H_*(X)$. **(15 marks)**

Solution:

(a) (i) The only homomorphism from A to B is the zero homomorphism [3]. Indeed, if $\phi\colon A \to B$ is a homomorphism then $\phi(A)$ is a subgroup of B and so has order dividing $|B|$. On the other hand, the First Isomorphism Theorem says that $|\phi(A)| = |A|/|\ker(\phi)|$, and this is a divisor of $|A|$. As $|A|$ and $|B|$ are coprime, we conclude that $|\phi(A)| = 1$, so $\phi(A) = \{0\}$, so $\phi = 0$. [7]

(ii) If $A \xrightarrow{f} U \xrightarrow{g} B$ is a short exact sequence, we claim that $U \simeq A \oplus B$ [3]. Indeed, we have $|U| = |A|.|B|$. We can write U as a direct sum of groups of the form $\mathbb{Z}/p^k$. As $|U| = |A|.|B|$ with $|A|$ and $|B|$ coprime, we see that p must divide $|A|$ or $|B|$ but not both. Let A' be the sum of all the factors where p divides $|A|$, and let B' be the sum of all the factors where p divides $|B|$, so $U = A' \oplus B'$. The homomorphism $f\colon A \to A' \oplus B'$ can be decomposed into a pair of homomorphisms $f_0\colon A \to A'$ and $f_1\colon A \to B'$. The homomorphism $g\colon A' \oplus B' \to B$ can be decomposed into a pair of homomorphisms $g_0\colon A' \to B$ and $g_1\colon B' \to B$. Here f_1 and g_0 are zero by part (i). As $f_1 = 0$ we have $\mathrm{img}(f) \leq A'$, and as $g_0 = 0$ we have $\ker(g) \geq A'$. As the sequence is exact we have $\mathrm{img}(f) = \ker(g)$, so this group must be equal to A'. Also, as f is injective we see that f_0 is injective, and as g is surjective we see that g_1 is surjective. It now follows that f_0 and g_1 are isomorphisms, and thus that $U = A' \oplus B; \simeq A \oplus B$ as claimed. [12]

(b) The connectivity assumptions mean that $H_0(X) = \mathbb{Z}$ and that we have a truncated Mayer-Vietoris sequence **[2]**. For $k > 1$ this takes the form

$$\mathbb{Z}/2^k \xrightarrow{e} \mathbb{Z}/3^k \oplus \mathbb{Z}/5^k \xrightarrow{f} H_k(X) \xrightarrow{g} \mathbb{Z}/2^{k-1} \xrightarrow{e} \mathbb{Z}/3^{k-1} \oplus \mathbb{Z}/5^{k-1}.\textbf{[3]}$$

The maps marked e are zero by (a)(i) **[2]**, so f is injective and g is surjective by exactness **[4]**, which means that the middle three terms form a short exact sequence. Thus, (a)(ii) tells us that

$$H_k(X) = \mathbb{Z}/3^k \oplus \mathbb{Z}/5^k \oplus \mathbb{Z}/2^{k-1} = \mathbb{Z}/(30^k/2)\textbf{[4]}$$

(where we have used the Chinese Remainder Theorem to tidy up the final answer a little). This formula remains valid for $k = 1$, although the argument is a tiny bit different.