MAS435: ALGEBRAIC TOPOLOGY
EXERCISES FOR WEEK 1 OF SEMESTER 2

To be handed in in the Thursday lecture of Week 2.

(a) Find a geometric simplicial complex K such that $|K|$ is the surface of the unit cube in $\mathbb{R}^3$. Draw a picture, and also give an explicit list of all the simplices. How many 0-simplices, 1-simplices and 2-simplices are there?

(b) Recall that I defined general position as follows: vectors $v_0, \dots, v_n \in \mathbb{R}^N$ are in general position if $f\colon \Delta^n \to \mathbb{R}^N$ is injective, where

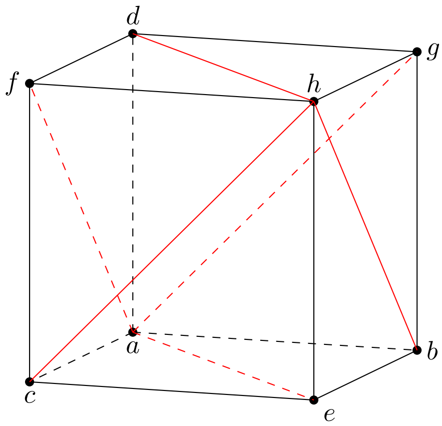
$$f(t_0, \dots, t_n) = t_0 v_0 + \dots + t_n v_n.$$

(Here $t \in \Delta^n$, so $t_i \geq 0$ for all i and $\sum_i t_i = 1$.) Show that if $v_1 - v_0, \dots, v_n - v_0$ are linearly independent, then $v_0, \dots, v_n$ are in general position. You can take $n = 3$ if you like.

(c) Write down all isomorphism classes of abelian groups of order 2, 4 and 8. Write down all isomorphism classes of abelian groups of order 6, 10, 15.

Solution:

(a) The picture is as follows:



The coordinates are

$$a = 000 \qquad b = 100 \qquad c = 010 \qquad d = 001$$
$$e = 110 \qquad f = 011 \qquad g = 101 \qquad h = 111.$$

The simplices are

$$\{a, b, c, d, e, f, g, h$$
$$ab, ac, ad, ae, af, ag, be, bg, bh, ce, cf, ch, df, dg, dh, eh, fh, gh,$$
$$abe, abg, ace, acf, adf, adg, beh, bgh, ceh, cfh, dfh, dgh\}.$$

There are eight 0-simplices: the vertices of the original cube. There are eighteen 1-simplices: the twelve edges of the cube, plus six extra edges, one cutting across the middle of each square face. There are twelve 2-simplices, two for each of the six square faces.

(b) Suppose that the vectors $v_1 - v_0, \dots, v_n - v_0$ are linearly independent. We must show that f is injective. Suppose that $s, t \in \Delta^n$ with $f(s) = f(t)$. We must show that $s = t$. Put $u = s - t \in \mathbb{R}^{n+1}$. We have $f(s) = f(t)$ so $\sum_{i=0}^n s_i v_i = \sum_{i=0}^n t_i v_i$ so $\sum_{i=0}^n u_i v_i = 0$. We also have $s, t \in \Delta^n$ so $\sum_{i=0}^n s_i = \sum_{i=0}^n t_i = 1$, so $\sum_{i=0}^n u_i = 1 - 1 = 0$. We can multiply this by the vector v_0 and subtract from the previous equation to get $\sum_{i=0}^n u_i(v_i - v_0) = 0$. The term for $i = 0$ is $u_0(v_0 - v_0) = 0$, so we can omit it. We now have $\sum_{i=1}^n u_i(v_i - v_0) = 0$. This is a linear relation between the vectors $v_1 - v_0, \dots, v_n - v_0$, which are assumed to be linearly independent, so the coefficients $u_1, \dots, u_n$ are all zero. We also have $\sum_{i=0}^n u_i = 0$, and it follows that the coefficient u_0 must be zero as well. We now have $u = 0$, but u was defined to be $s - t$, so $s = t$ as required.

(c) Any finite abelian group can be expressed as a direct sum of terms $\mathbb{Z}/p^k$ with p prime and $k > 0$. From this we obtain the following lists:

- Order 2: $\mathbb{Z}/2$
- Order 4: $\mathbb{Z}/4$, $\mathbb{Z}/2 \oplus \mathbb{Z}/2$.
- Order 8: $\mathbb{Z}/8$, $\mathbb{Z}/4 \oplus \mathbb{Z}/2$, $\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

Next, if p and q are distinct primes, the only way to make an abelian group of order pq is $\mathbb{Z}/p \oplus \mathbb{Z}/q$. (You might ask about $\mathbb{Z}/pq$, but that is isomorphic to $\mathbb{Z}/p \oplus \mathbb{Z}/q$, by the Chinese Remainder Theorem.) Thus, for orders 6, 10 and 15 we just have $\mathbb{Z}/2 \oplus \mathbb{Z}/3$, $\mathbb{Z}/2 \oplus \mathbb{Z}/5$ and $\mathbb{Z}/3 \oplus \mathbb{Z}/5$.