QUIZZES FOR MAS435 ALGEBRAIC TOPOLOGY

As the semester progresses, new quizzes and solutions will be added to this file.

Week 1

(a) Give an example of a space X such that $\pi_1(X)$ is nontrivial, finite and abelian.
(b) Give an example of a space Y such that $\pi_1(Y)$ is infinite and nonabelian.

Solution:

The most obvious possibilities are $X = \mathbb{R}P^2$ and $Y = S^1 \vee S^1$. In more detail, the main examples that have been covered are as follows:

(1) If Z is contractible then $\pi_1(Z)$ is trivial. The trivial group is finite and abelian, but part (a) specifies that we want a nontrivial group. So a contractible space is not valid for (a) or (b).
(2) $\pi_1(S^1)$ is $\mathbb{Z}$, which is nontrivial, infinite and abelian, so again this is not valid for (a) or (b). (The proof that $\pi_1(S^1) = \mathbb{Z}$ uses the covering map $f\colon \mathbb{R} \to S^1$ given by $f(t) = (\cos(t), \sin(t))$.)
(3) For the torus $T = S^1 \times S^1$ we have $\pi_1(T) = \mathbb{Z} \times \mathbb{Z}$, which is nontrivial, infinite and abelian, so again this is not valid for (a) or (b). (Here we have just used (2) together with the general result that $\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$.)
(4) For the sphere S^n with $n > 1$, the fundamental group $\pi_1(S^n)$ is trivial, which is again not useful. (This can be proved by dividing S^n into two hemispheres, and using the van Kampen Theorem.)
(5) For the real projective space $\mathbb{R}P^n$ with $n > 1$ we have $\pi_1(\mathbb{R}P^n) = \mathbb{Z}/2$, so this is valid for (a). This can be proved using the double covering map $S^n \to \mathbb{R}P^n$. The case $n = 1$ is exceptional: $\mathbb{R}P^1$ is homeomorphic to S^1, so $\pi_1(\mathbb{R}P^1) = \mathbb{Z}$.
(6) For the figure eight space E, the fundamental group has two generators a and b with no relations, so in particular $ab \neq ba$ and the powers a^n are all different. This means that $\pi_1(E)$ is infinite and nonabelian, so it is valid for (b).
(7) For the Klein bottle K, the fundamental group has two generators a and b subject only to the relation $aba^{-1} = b^{-1}$. We again see that a and b do not commute and the powers of a are all different, so again we have an infinite nonabelian group, as required for (b).
(8) If Z is a surface of genus g, then $\pi_1(Z)$ has generators $a_1, \ldots, a_g, b_1, \ldots, b_g$ subject only to the relation
$$a_1 b_1 a_1^{-1} b^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = 1.$$
In the case $g = 1$ the relation is just $a_1 b_1 a_1^{-1} b_1^{-1} = 1$, which means that $a_1 b_1 = b_1 a_1$, so a_1 and b_1 commute. In fact, we find that $\pi_1(Z)$ is a copy of $\mathbb{Z}^2$, as we expect because a surface of genus 1 is just a torus. However, if $g > 1$ then $\pi_1(Z)$ is infinite and nonabelian, so we have another valid answer for (b).

A couple of people offered answers where X or Y is a disjoint union of two pieces, for example $X = S^1 \amalg S^1$. Examples of this type will never be useful for this kind of problem. If $X = U \amalg V$ then $\pi_1(X)$ is just $\pi_1(U)$ (if the basepoint lies in U) or $\pi_1(V)$ (if the basepoint lies in V). This is because a continuous loop that starts in U can never cross over to V.

Week 2

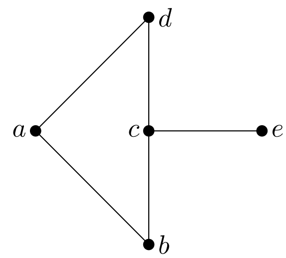
(a) Let K be a simplicial complex that includes at least one 2-simplex. What is the minimum possible total number of simplices in K?
(b) Give an example of a space that has no triangulation.

Solution:

(a) If we have a 2-simplex then we must also have the three edges and three vertices of that 2-simplex, making seven simplices in total. We can just take K to consist of the standard 2-simplex in $\mathbb{R}^3$ together with all of its faces; in that case, we get exactly seven simplices.

(b) The most obvious example is that $\mathbb{R}$ has no triangulation. Indeed, if X has a triangulation then it is homeomorphic to $|K|$ for some K, and $|K|$ is always compact, so X must be compact. As $\mathbb{R}$ is not compact, it cannot be triangulated. The other example that we discussed in lectures is the space $X = \{0, 1, 1/2, 1/3, 1/4, \dots\}$, which is compact and Hausdorff but which has no triangulation. Roughly speaking, this is the simplest example of a space with "small scale fine structure", and such spaces typically cannot be triangulated. Fractals provide more elaborate examples.

Some people offered a space X like this:



It is true that the set

$$K = \{\langle a,b\rangle, \langle a,d\rangle, \langle b,d\rangle, \langle c,e\rangle, \langle a\rangle, \langle b\rangle, \langle c\rangle, \langle d\rangle, \langle e\rangle\}$$

is not a simplicial complex, but this is not relevant. The set

$$L = \{\langle a,b\rangle, \langle a,d\rangle, \langle b,c\rangle, \langle c,d\rangle, \langle c,e\rangle, \langle a\rangle, \langle b\rangle, \langle c\rangle, \langle d\rangle, \langle e\rangle\}$$

is a perfectly good simplicial complex with $|L| = X$, so X has a triangulation.

Some other people offered examples like the torus $T = S^1 \times S^1$ or the real projective plane $\mathbb{R}P^2$. Because these are curved, they cannot actually be equal to $|K|$ for any simplicial complex K. However, for a triangulation X just has to be homeomorphic to $|K|$, not equal to it. It takes some work to produce triangulations of T and $\mathbb{R}P^2$, but it can be done.

Week 3

(a) Let $P \xrightarrow{f} Q \xrightarrow{g} R$ be homomorphisms of abelian groups. What does it mean to say that this sequence is *exact*? What about *short exact*?

(b) Does there exist a short exact sequence $\mathbb{Z}/10 \xrightarrow{p} \mathbb{Z}/100 \xrightarrow{q} \mathbb{Z}/1000$?

(c) Does there exist an exact sequence $\mathbb{Z}/10 \xrightarrow{u} \mathbb{Z} \xrightarrow{v} \mathbb{Z}$?

Solution:

(a) Exactness means that $\text{image}(f) = \ker(g)$. The sequence is short exact if, in addition, f is injective and g is surjective. Equivalently, the sequence $P \xrightarrow{f} Q \xrightarrow{g} R$ is short exact iff every three-term subsequence of $0 \to P \xrightarrow{f} Q \xrightarrow{g} R \to 0$ is exact.

(b) In a short exact sequence $A \to B \to C$ of finite abelian groups, we have $|B| = |A|.|C|$. As $100 \neq 10.1000$, there can be no short exact sequence $\mathbb{Z}/10 \xrightarrow{p} \mathbb{Z}/100 \xrightarrow{q} \mathbb{Z}/1000$. Alternatively, we can just say that $|\mathbb{Z}/1000| > |\mathbb{Z}/100|$, so there cannot be any surjective map from $\mathbb{Z}/100$ to $\mathbb{Z}/1000$.

(c) The sequence $\mathbb{Z}/10 \xrightarrow{0} \mathbb{Z} \xrightarrow{1} \mathbb{Z}$ is exact. (It is not short exact, but the question did not ask for that.)

A couple of people gave answers involving the inclusions $i\colon \mathbb{Z}/10 \to \mathbb{Z}/100$ or $j\colon \mathbb{Z}/10 \to \mathbb{Z}$. Presumably the intent is that for $a = 0, \dots, 9$ we define $i(a \bmod 10) = a \bmod 100$ and $j(a \bmod 10) = a$. The problem is that these maps are not homomorphisms. For example, in $\mathbb{Z}/10$ we have $4+6 = 0$, but $j(4)+j(6) = 10 \neq j(0)$. In fact, if $f\colon \mathbb{Z}/10 \to \mathbb{Z}$ is a homorphism then for all $x \in \mathbb{Z}/10$ we have $10\,f(x) = f(10x) = f(0) = 0$. As $f(x)$ is just an integer, the only way we can have $10\,f(x) = 0$ is if $f(x) = 0$. This shows that the only homomorphism from $\mathbb{Z}/10$ to $\mathbb{Z}$ is zero. There are some injective homomorphisms from $\mathbb{Z}/10$ to $\mathbb{Z}/100$, such as $g(a \bmod 10) = 10a \bmod 100$ or $h(a \bmod 10) = 70a \bmod 100$. However, none of these give a short exact sequence of the required type.