MAS435 ALGEBRAIC TOPOLOGY — PROBLEM SHEET 1

Please hand in Exercises 1, 3 and 6 before the Tuesday lecture of Week 3.
Your answers should be submitted via Blackboard. Please either

(a) Use LaTeX, and upload the resulting PDF file; or
(b) Scan handwritten work with a scanning app such as CamScanner (Android/ Apple) or Microsoft Lens (Android/ Apple) or Adobe Scan (Android/ Apple). Again, you should upload files in PDF form.

Definition. Let Q be a finite subset of a metric space. Fix a number $\epsilon \geq 0$, and declare that a subset $\sigma \subseteq Q$ is a simplex iff $\text{diam}(\sigma) \leq \epsilon$, where

$$\text{diam}(\sigma) = \max\{d(a, b) \mid a, b \in \sigma\}.$$

This gives an abstract simplicial complex $\text{VR}_\epsilon(Q)$ with vertex set Q, called the *Vietoris-Rips complex*.

Definition. Let X be a set, and let $\underline{U} = (U_1, \ldots, U_n)$ be a list of nonempty subsets of X. We declare that a nonempty subset

$$\sigma = \{i_0, \ldots, i_d\} \subseteq \{1, \ldots, U_n\}$$

is a simplex iff $U_{i_0} \cap \cdots \cap U_{i_d} \neq \emptyset$. This gives a simplicial complex $\check{C}(\underline{U})$ with vertex set $\{1, \ldots, n\}$, called the *Čech nerve* of $\underline{U}$. (This is most often considered in the case where $X = U_1 \cup \cdots \cup U_n$, so the sets U_i cover X.)

Definition. Let G be a one-dimensional simplicial complex with vertex set V, so it has only vertices and edges, but no higher simplices. We define a new simplicial complex $\text{Flag}(G)$ by declaring that $\sigma \subseteq V$ is a simplex iff $\{a, b\}$ is a simplex of G for all $a, b \in V$. This is called the *flag complex* of G.

Exercise 1. Draw sketches or diagrams to illustrate the following spaces: $S^0 \times S^2$, $S^1 \times [0, 1]$, $S^1 \times S^1$ and $S^2 \times [0, 1]$.

Exercise 2. Let a be the point $(2, 0) \in \mathbb{R}^2$. Draw pictures of the following subsets of $\mathbb{R}^2$.

$$X_1 = \{u \in \mathbb{R}^2 \mid d_1(0, u) = 1\} \qquad X_2 = \{u \in \mathbb{R}^2 \mid d_2(0, u) = 1\}$$
$$X_3 = \{u \in \mathbb{R}^2 \mid d_\infty(0, u) \leq 1\} \qquad X_4 = \{u \in \mathbb{R}^2 \mid d_2(u, 0) = d_2(u, a)\}$$
$$X_5 = \{u \in \mathbb{R}^2 \mid d_2(u, 0) = d_2(u, a) = 1\} \qquad X_6 = \{u \in \mathbb{R}^2 \mid d_2(0, a) = d_2(0, u) + d_2(u, a)\}$$
$$X_7 = \{u \in \mathbb{R}^2 \mid d_2(0, a) \leq d_2(0, u) + d_2(u, a)\}$$

Hint: particularly in the last four parts, think geometrically.

Exercise 3. Find a geometric simplicial complex K such that $|K|$ is the surface of the unit cube in $\mathbb{R}^3$. Draw a picture, and also give an explicit list of all the simplices. How many 0-simplices, 1-simplices and 2-simplices are there?

Exercise 4. Describe $\text{VR}_\epsilon(Q)$ (as defined at the top of this sheet) in the case where Q consists of the four corners of the unit square. (The answer depends on the value of ϵ, in a way which you should describe.)

Exercise 5. Let V be the set of 2-element subsets of $\{1, 2, 3, 4\}$. Define G to be the 1-dimensional simplicial complex with edges consisting of elements of V with an element in common. Draw G, then draw $\text{Flag}(G)$. Finally, let K be the Čech nerve (as defined at the top of this sheet) of the set of two element subsets of $\{1, 2, 3, 4\}$. Draw K. Is K a flag complex?

Exercise 6. Let K be an abstract simplicial complex, and let σ be a simplex of K. We define a new simplicial complex called the *link* of σ as follows:

$$\text{vert}(\text{link}(\sigma, K)) = \{v \in \text{vert}(K) \setminus \sigma \mid \{v\} \cup \sigma \in \text{simp}(K)\}$$
$$\text{simp}(\text{link}(\sigma, K)) = \{\tau \in \text{simp}(K) \mid \tau \cap \sigma = \emptyset \text{ and } \tau \cup \sigma \in \text{simp}(K)\}.$$

Show that this does indeed give a simplicial complex. Let K be the octahedron, and draw a picture of the link of σ when σ is (i) a vertex (ii) an edge and (iii) a 2-simplex.