MAS435 ALGEBRAIC TOPOLOGY — PROBLEM SHEET 2

Please hand in Exercises 1 and 2 before the Tuesday lecture of Week 4.
Your answers should be submitted via Blackboard. Please either

(a) Use LaTeX, and upload the resulting PDF file; or
(b) Scan handwritten work with a scanning app such as CamScanner (Android/ Apple) or Microsoft Lens (Android/ Apple) or Adobe Scan (Android/ Apple). Again, you should upload files in PDF form.

Exercise 1.

(a) Let X be any set, and let U be a subset with $\emptyset \neq U \neq X$. Show that there is a topology on X for which $\emptyset$, U and X are the only open sets.
(b) Suppose that $|X| = 2$, say $X = \{a, b\}$. Describe all possible topologies on X.
(c) Suppose instead that $|X| = 3$, say $X = \{a, b, c\}$. For a topology on X, what is the minmum possible number of open sets? What is the maximum possible number? How many topologies are there with precisely three open sets? Can you find a topology with precisely four open sets? (There are in fact 29 different topologies on X. If you are enthusiastic, you can try to classify them all.)

Exercise 2. Which of the following rules gives a well-defined, continuous function $f: X \to Y$? Justify your answer with either a geometric or an algebraic argument.

(a) X is the unit circle centred at the origin in $\mathbb{R}^2$, Y is the circle of radius 3 centred at $(2, 0)$, and $f(x, y)$ is the point where the ray from $(0, 0)$ outwards through (x, y) meets Y.
(b) X is the unit circle centred at the origin in $\mathbb{R}^2$, Y is the line $x = 2$, and $f(x, y)$ is the point where the line through $(0, 0)$ and (x, y) meets Y.
(c) X is the northern hemisphere, Y is the equator, and $f(a)$ is the point where the shortest route along the surface of the earth from a to the south pole crosses Y.
(d) X is the unit circle centred at $(0, 2)$, Y is the x-axis, and $f(a)$ is the point where the vertical line through a crosses Y.

Exercise 3. Which of the following sets in the plane are (i) open, (ii) closed or (iii) neither open nor closed.

(a) the set $A = [0, 1] \times [0, 1)$.
(b) the real axis.
(c) the upper half-plane $B = \{(x, y) \mid y > 0\}$.
(d) the set of points $C = \{(x, y) \mid x \in \mathbb{Q}\}$.
(e) the set of points $D = \{\theta e^{i\theta} \mid \theta \geq 0\}$.

Exercise 4. Let X be a topological space, let $U \subseteq X$ be open and let $F \subseteq X$ be closed, and suppose that $X = U \cup F$. Suppose that $f: U \to Y$ and $g: F \to Y$ are continuous maps such that $f(x) = g(x)$ for all $x \in U \cap F$, so there is a unique function $h: X \to Y$ such that $h(x) = f(x)$ for $x \in U$ and $h(x) = g(x)$ for $x \in F$. Give an example to show that h need not be continuous.

Exercise 5. Let $GL_2(\mathbb{R})$ be the space of all invertible 2×2 matrices over $\mathbb{R}$. Define a function $\xi: GL_2(\mathbb{R}) \to GL_2(\mathbb{R})$ by $\xi(A) = A^{-1}$. Is this continuous?

Exercise 6. Define a function $\phi: M_2(\mathbb{R}) \to M_2(\mathbb{R})$ by

$$\phi(A) = \text{ the row-reduced echelon form of } A,$$

so for example

$$\phi\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \phi\begin{bmatrix} 4 & 8 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}.$$

Is ϕ a continuous map?