MAS435 ALGEBRAIC TOPOLOGY — PROBLEM SHEET 3

Please hand in Exercises 1 and 3 by the Friday lecture of Week 5.
Your answers should be submitted via Blackboard. Please either

(a) Use LaTeX, and upload the resulting PDF file; or
(b) Scan handwritten work with a scanning app such as CamScanner (Android/ Apple) or Microsoft Lens (Android/ Apple) or Adobe Scan (Android/ Apple). Again, you should upload files in PDF form.

Exercise 1. Put $SL_2(\mathbb{R}) = \{A \in M_2(\mathbb{R}) \mid \det(A) = 1\}$. This evidently contains the group $SO(2)$, which we have seen is homeomorphic to a circle. Here we will show that $SL_2(\mathbb{R})$ is homeomorphic to $S^1 \times \mathbb{R}^+ \times \mathbb{R}$ (where $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\} = (0, \infty)$).

Given $(x, y) \in S^1$ and $a \in \mathbb{R}^+$ and $b \in \mathbb{R}$ we put

$$R(x,y) = \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \qquad D(a) = \begin{bmatrix} a & 0 \\ 0 & 1/a \end{bmatrix}$$

$$T(b) = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \qquad f(x,y,a,b) = R(x,y)D(a)T(b).$$

(a) Check that all the above matrices lie in $SL_2(\mathbb{R})$, so we have a continuous map $f\colon S^1 \times \mathbb{R}^+ \times \mathbb{R} \to SL_2(\mathbb{R})$. Give a more explicit formula for f.
(b) Suppose that $B = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \in SL_2(\mathbb{R})$. Prove that $p^2 + r^2 > 0$. Deduce that there is a continuous map $g\colon SL_2(\mathbb{R}) \to \mathbb{R}^4$ given by

$$g\begin{bmatrix} p & q \\ r & s \end{bmatrix} = \left(\frac{p}{\sqrt{p^2+r^2}}, \frac{r}{\sqrt{p^2+r^2}}, \sqrt{p^2+r^2}, \frac{pq+rs}{p^2+r^2} \right).$$

(c) Prove that for all $B \in SL_2(\mathbb{R})$ we have $g(B) \in S^1 \times \mathbb{R}^+ \times \mathbb{R}$, so we can regard g as a continuous map $SL_2(\mathbb{R}) \to S^1 \times \mathbb{R}^+ \times \mathbb{R}$.
(d) Prove that for $(x, y, a, b) \in S^1 \times \mathbb{R}^+ \times \mathbb{R}$ we have $g(f(x, y, a, b)) = (x, y, a, b)$.
(e) Prove that for $B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ we have $prs - r = qr^2$ and $pqr + p = sp^2$ and $f(g(B)) = B$.
(f) Deduce that f and g are homeomorphisms.

Exercise 2. (a) Let X be the union of the three axes in $\mathbb{R}^3$, and put $Y = \{z \in \mathbb{C} \mid z^3 \in \mathbb{R}\}$. Explain using pictures why X is homeomorphic to Y.
(b) Let X be the union of the twelve edges of a cube. Draw a subset of $\mathbb{R}^2$ that is homeomorphic to X.
(c) Take a long straight pipe, with walls of zero thickness. Cut a small hole in the side of the pipe and call the resulting surface X. Draw a subset of $\mathbb{R}^2$ that is homeomorphic to X.
(d) Put $X = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$ and $Y = \{z \in \mathbb{C} \mid 1 < |z| < 2\}$. Give a formula for a homeomorphism $f\colon X \to Y$.

Exercise 3. For each of the following spaces X, calculate $\pi_0(X)$. For the first three cases, give careful justifications. For the remaining cases you can just give an answer and (optionally) a picture.

$$X_1 = \mathbb{R} \setminus \{0\} \qquad X_2 = S^2 \qquad X_3 = \mathbb{Q}$$

$$X_4 = \mathbb{R} \setminus \{0, \ldots, n-1\} \qquad X_5 = \mathbb{R}^2 \setminus \{(0,0)\} \qquad X_6 = \mathbb{R}^2 \setminus \{(0,y) \mid y \in \mathbb{R}\}$$

$$X_7 = \{(x,y) \in \mathbb{R}^2 \mid xy = 0,\ x + y \neq 0\} \qquad X_8 = S^1 \times S^1 \qquad X_9 = \mathbb{R} \setminus \mathbb{Z}$$

Exercise 4. Let a, b, c, d be points in a space X, and suppose we have paths u, v, w in X with u running from a to b, v from b to c, and w from c to d.

Define $z\colon [0,3] \to X$ by

$$z(t) = \begin{cases} 0 \leq t \leq 1 & u(t) \\ 1 \leq t \leq 2 & v(t-1) \\ 2 \leq t \leq 3 & w(t-2). \end{cases}$$

Find functions $f, g\colon I \to [0,3]$ such that $((u * v) * w) = z \circ f$ and $(u * (v * w)) = z \circ g$, and plot their graphs. Using this, give an alternative proof that $u * (v * w)$ is path homotopic to $(u * v) * w$.

Exercise 5. Put $X = \{(x,y) \in \mathbb{R}^2 \mid y^2 = x^3 - x\}$. (This is an example of an *elliptic curve*; such curves are important in number theory and certain other areas of mathematics.)

Sketch the graph of the function $x^3 - x$, then draw a picture of X. By considering the function $f\colon X \to \mathbb{R}$ given by $f(x,y) = 2x - 1$, show that X is not path-connected.

Exercise 6. Recall that for any space X and any $x \in X$ we have a constant path $c_x\colon I \to X$ given by $c_x(t) = x$ for all t.

(a) Let $u\colon I \to X$ be a path. Prove that u is homotopic to $c_{u(0)}$.
(b) Suppose that X is path-connected. Prove that any two paths $u, v\colon I \to X$ are homotopic to each other.

Exercise 7. In this problem, we'll show that the space $GL_n(\mathbb{C})$ (of invertible $n \times n$ matrices over the complex numbers) is path-connected.

Let A be an $n \times n$ invertible complex matrix, with eigenvalues $\lambda_1, \dots, \lambda_r$ say. Because A is invertible, we know that $\lambda_i \neq 0$ for all i. Define

$$L_i = \{-t\lambda_i \mid 0 \leq t < \infty\} \subset \mathbb{C},$$

so L_i is the half-line starting at 0 and passing through $-\lambda_i$.

(1) Show that the eigenvalues of $\alpha A + \beta I$ are $\alpha\lambda_1 + \beta, \dots, \alpha\lambda_r + \beta$.
(2) Let μ be a nonzero complex number. Show that the linear path from λ_i to μ passes through 0 iff $\mu \in L_i$ (think geometrically).
(3) Show that the linear path from A to μI in $M_n(\mathbb{C})$ actually lies in $GL_n(\mathbb{C})$ iff $\mu \notin L_i$ for all i.
(4) Deduce that A can be connected in $GL_n(\mathbb{C})$ to some matrix of the form μI with $\mu \neq 0$.
(5) Show that all matrices of the form μI can be connected to I in $GL_n(\mathbb{C})$.
(6) Deduce that $GL_n(\mathbb{C})$ is connected.

Exercise 8. Let X be the set of all functions $f\colon [-1,1] \to \mathbb{R}$ that satisfy the differential equation $f' + f = 0$, considered as a metric space using the max-metric. Describe the elements of X more explicitly, and give a simple formula for the metric. Let Y be the subspace of X consisting of those functions $f \in X$ for which $f(1)f(-1) = 1$. Show that X is path-connected but Y is not.

Exercise 9. Define

$$X = \{A \in M_2(\mathbb{R}) \mid A^2 = A\}.$$

(1) If $A \in X$, what can you say about $\det(A)$?
(2) Deduce that X is not connected.
(3) Show that if $A \in X$ and A is invertible then $A = I$.
(4) Show that if $A \in X$ then $I - A \in X$.
(5) Show that if $A \in X$ and $\det(I - A) = 1$ then $A = 0$.
(6) Show that the path-component of I in X is just $\{I\}$, and the path-component of 0 is just $\{0\}$.