MAS435 ALGEBRAIC TOPOLOGY — PROBLEM SHEET 4

Please hand in Exercises 1 and 3 by the Tuesday lecture of Week 6.
Your answers should be submitted via Blackboard. Please either

(a) Use LaTeX, and upload the resulting PDF file; or
(b) Scan handwritten work with a scanning app such as CamScanner (Android/ Apple) or Microsoft Lens (Android/ Apple) or Adobe Scan (Android/ Apple). Again, you should upload files in PDF form.

Exercise 1. For any set X, put $FX = X \times X$. Explain how to define Ff so as to make this into a functor $F\colon \mathrm{Set} \to \mathrm{Set}$.

Exercise 2. For any group G, we put

$$ZG = \text{ center of } G = \{z \in G \mid zg = gz \text{ for all } g \in G\}.$$

Consider the cyclic group $C_2 = \{1, -1\}$ and the permutation group S_3. Prove that C_2 is a retract of S_3 but ZC_2 is not a retract of ZS_3. Deduce that there is no way to make Z into a functor Group $\to$ Group.

Exercise 3. For any group G, put $TG = \{g \in G \mid g^2 = 1\}$. Explain how to define $T\phi\colon TG \to TH$ for every homomorphism $\phi\colon G \to H$, so as to make T into a functor Group $\to$ Set. By considering some small groups G, explain why it is not a functor Group $\to$ Group.

Exercise 4. Define two different functors $D, I\colon \mathrm{Set} \to \mathrm{Top}$, such that DX is just the set X equipped with a certain topology, and IX is just the set X equipped with a different topology.

Exercise 5. Define functors $D, E\colon \mathrm{Set} \to \mathrm{Metric}_1$ such that D is interesting but E is as trivial as possible. (Hint: find a metric that gives rise to the discrete topology.)

Exercise 6. Let X be a path connected space, and let Y be a retract of X. Prove that Y is also path connected.

Proof. As X is path connected, we have $|\pi_0(X)| = 1$, so $\pi_0(X) = \{a\}$ for some a. As Y is a retract of X, there are continuous maps $f\colon Y \to X$ and $g\colon X \to Y$ with $g \circ f = \mathrm{id}$. As π_0 is a functor, we see that $g_* \circ f_* = \mathrm{id}\colon \pi_0(Y) \to \pi_0(Y)$. In other words, for any $b \in \pi_0(Y)$ we have $b = g_*(f_*(b))$. Here $f_*(b)$ is an element of the set $\pi_0(X) = \{a\}$, so we must have $f_*(b) = a$, so the relation $b = g_*(f_*(b))$ can be rewritten as $b = g_*(a)$. We now see that $\pi_0(Y) = \{g_*(a)\}$. As this is a singleton, it follows that Y is path connected. □