MAS435 ALGEBRAIC TOPOLOGY — PROBLEM SHEET 5

Please hand in Exercises 1 and 3 by the Tuesday lecture of Week 8.
Your answers should be submitted via Blackboard. Please either

(a) Use LaTeX, and upload the resulting PDF file; or
(b) Scan handwritten work with a scanning app such as CamScanner (Android/ Apple) or Microsoft Lens (Android/ Apple) or Adobe Scan (Android/ Apple). Again, you should upload files in PDF form.

Exercise 1. Consider the following spaces:

$$\begin{aligned} X_0 &= \text{ the union of all the edges of a cube} \\ X_1 &= \{(x,y) \in [-1,1] \times [-1,1] \mid x \in \mathbb{Z} \text{ or } y \in \mathbb{Z}\} \\ X_2 &= \{(x,y,z) \in S^2 \mid xyz = 0\} \end{aligned}$$

Draw pictures of these spaces (or of subsets of $\mathbb{R}^2$ that are homeomorphic to them). Evaluate $a(X_i)$ for $i = 0, 1, 2$, and deduce that none of the spaces are homeomorphic to each other.

Exercise 2. Let Y and Z be disjoint topological spaces, and put $X = Y \cup Z$, and give this the coproduct topology. Consider another topological space T and a function $p\colon T \to X$. Put $A = p^{-1}(Y) \subseteq T$ and $B = p^{-1}(Z) \subseteq T$. Prove that p is continuous iff the following conditions are satisfied:

(1) A and B are open subsets of T.
(2) The restricted map $p|_A\colon A \to Y$ is continuous (with respect to the subspace topology on A and the originally given topology on Y).
(3) The restricted map $p|_B\colon B \to Z$ is continuous (with respect to the subspace topology on B and the originally given topology on Z).

(This was stated without proof as Proposition 8.5 in the notes.)

Exercise 3. Let Y and Z be metric spaces. Define two different metrics on $Y \times Z$ by

$$\begin{aligned} d_\infty((y,z),(y',z')) &= \max(d(y,y'), d(z,z')) \\ d_1((y,z),(y',z')) &= d(y,y') + d(z,z'). \end{aligned}$$

Show that in both cases, the resulting metric topology is the same as the product topology.

Exercise 4. Define a relation on S^n by xEy iff ($x = y$ or $x = -y$). This is easily seen to be an equivalence relation, so we can define $\mathbb{R}P^n = S^n/E$. (This is called *real projective space* of dimension n.) We also define

$$P_n = \{A \in M_{n+1}(\mathbb{R}) \mid A^2 = A^T = A, \quad \text{trace}(A) = 1\}.$$

(a) Prove that P_n is a closed subset of $M_{n+1}(\mathbb{R})$.
(b) Show that there is a continuous map $f\colon S^n \to P_n$ given by $f(u)_{ij} = u_i u_j$ for all $u \in S^n$. Show also that $f(u)v = \langle u, v\rangle u$ for all $v \in \mathbb{R}^{n+1}$.
(c) Show that f is E-saturated, and so gives a continuous map $\overline{f}\colon \mathbb{R}P^n \to P_n$.
(d) Now suppose that $A \in P_n$. Prove that $\mathbb{R}^{n+1} = \ker(A) \oplus \text{img}(A)$, where $\ker(A) = \{v \mid Av = 0\}$ and $\text{img}(A) = \{Av \mid v \in \mathbb{R}^{n+1}\}$. Prove also that if $t \in \ker(A)$ and $u \in \text{img}(A)$ then the inner product $\langle t, u\rangle$ is zero.
(e) Prove also that $\dim(\text{img}(A)) = 1$.
(f) If u is a unit vector in $\text{img}(A)$ then $A = f(u)$.
(g) Deduce that the map $\overline{f}\colon \mathbb{R}P^n \to P_n$ is a continuous bijection. (In fact it is a homeomorphism, but we will defer the proof of that.)