MAS435 ALGEBRAIC TOPOLOGY — PROBLEM SHEET 6

Please hand in Exercises 1 and 2 by the Tuesday lecture of Week 9.
Your answers should be submitted via Blackboard. Please either

(a) Use LaTeX, and upload the resulting PDF file; or
(b) Scan handwritten work with a scanning app such as CamScanner (Android/ Apple) or Microsoft Lens (Android/ Apple) or Adobe Scan (Android/ Apple). Again, you should upload files in PDF form.

Exercise 1. Define a continuous map $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = \exp(-x^2)$.

(a) Sketch the graph of f.
(b) Find an open set $U \subseteq \mathbb{R}$ such that $f(U)$ is not open.
(c) Find a closed set $F \subseteq \mathbb{R}$ such that $f(F)$ is not closed.
(d) Find a compact set $K \subseteq \mathbb{R}$ such that $f^{-1}(K)$ is not compact.
(e) Find sets $A, B \subseteq \mathbb{R}$ such that $f(A \cap B) \neq f(A) \cap f(B)$.

Exercise 2. Which of the following subsets of $\mathbb{C}$ are compact?

(a) $A = \{z \in \mathbb{C} \mid f(z) = 0\}$ (where f is a fixed polynomial of degree $d > 0$).
(b) $B = \{z \in \mathbb{C} \mid \sin(z) = 0\}$.
(b) $C = \{x + iy \in \mathbb{C} \mid 0 \leq x \leq 1\}$.
(d) $D = \{z \in \mathbb{C} \mid |z| \leq 1\}$.
(e) $E = \{z \in \mathbb{C} \mid |z^{10} + z| \leq 1\}$. [Hint: give a lower bound for $|z^{10} + z|$ when $|z| \geq 2$ say.]
(f) $F = \{z \in \mathbb{C} \mid z^n = 1 \text{ for some } n > 0\}$.

Exercise 3. Give an example of a topological space X and a subspace Y that is compact but not closed.

Exercise 4. Consider the space $X = \{0\} \cup \{1/n \mid n \geq 1\}$. Prove that this is not homeomorphic to $\mathbb{Z}$.

Exercise 5. For $n \geq 1$, let X_n be the circle in $\mathbb{R}^2$ of radius $1/n$ centred at $(1/n, 0)$. Let X be the union of all the spaces X_n (this is sometimes called the *Hawaiian earrings*). Draw a picture of X, and prove that it is compact.

Exercise 6. (a) Give an example of a space X, and a continuous map $f\colon X \to \mathbb{R}$ such that $f(x) > 0$ for all $x \in X$ but there is no $\epsilon > 0$ such that $f(x) \geq \epsilon$ for all $x \in X$.
(b) Suppose that X is compact and that $f\colon X \to \mathbb{R}$ is continuous and $f(x) > 0$ for all x. Show that there is an $\epsilon > 0$ such that $f(x) \geq \epsilon$ for all x.

Exercise 7. For each of the following pairs of spaces, either give a continuous surjective map $f\colon X \to Y$, or prove that no such map exists.

(a) $X = \mathbb{R}$ and $Y = S^1$
(b) $X = \mathbb{C}$ and $Y = \mathbb{C} \setminus \{0\}$
(c) $X = \mathbb{R}$ and $Y = \mathbb{R} \setminus \{0\}$
(d) $X = S^2$ and $Y = S^2 \setminus \{\text{north pole}\}$
(e) $X = S^2$ and $Y = S^1$.

Exercise 8. Let X be a compact subset of $\mathbb{R}^n$, and let x be a point of X.

(a) Give an example of this situation where $X \setminus \{x\}$ is not compact.
(b) Give an example of this situation where $X \setminus \{x\}$ is compact.
(c) Find a general rule which predicts when $X \setminus \{x\}$ is compact.