MAS435 ALGEBRAIC TOPOLOGY — PROBLEM SHEET 7

Please hand in Exercises 1 and 2 by the Tuesday lecture of Week 10.
Your answers should be submitted via Blackboard. Please either

(a) Use LaTeX, and upload the resulting PDF file; or
(b) Scan handwritten work with a scanning app such as CamScanner (Android/ Apple) or Microsoft Lens (Android/ Apple) or Adobe Scan (Android/ Apple). Again, you should upload files in PDF form.

Exercise 1. Put

$$X = \{z \in \mathbb{C} \mid 1 < |z| < 5\} \qquad\qquad Y = S^1 = \{z \in \mathbb{C} \mid |z| = 1\}.$$

Define maps $f\colon X \to Y$ and $g_0, g_1\colon Y \to X$ by

$$f(z) = z/|z| \qquad\qquad g_0(z) = 3z \qquad\qquad g_1(z) = 3 + z$$

Draw pictures.

Are the following statements true, false, or meaningless? Justify your answers.

(a) f is homotopic to g_0
(b) fg_0 is homotopic to 1_X
(c) fg_0 is homotopic to 1_Y
(d) $g_0 f$ is homotopic to 1_X
(e) $g_1 f$ is homotopic to 1_X

Exercise 2. Show that $\mathbb{R}^3 \setminus S^2$ is homotopy equivalent to $S^2 \cup \{0\}$.

Exercise 3. Let S_n be the set of permutations of $\{1, \ldots, n\}$. We can identify a permutation π with the vector $(\pi(1), \ldots, \pi(n))$ and thus think of S_n as a finite subset of $\mathbb{R}^n$. Define

$$F_n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \neq x_j \text{ for all } i \neq j\}.$$

Prove that F_n is homotopy equivalent to S_n. (Draw the case $n = 2$ first.)

Exercise 4. Let X be a topological space, and let $f, g\colon X \to S^1$ be continuous maps. Suppose that f is linearly homotopic to g. What can you deduce?

Exercise 5. Define $f\colon \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = \cos(\sqrt{x^2 + y^2})$. Let X be the set of points where $f \neq 0$, and let Y be the set of points where $\partial f/\partial r = 0$. (The value of $\partial f/\partial r$ at $(0, 0)$ is not really well-defined, but we will take it to be 0.) Prove that X is homotopy equivalent to Y.

Exercise 6. Put $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Consider another matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2\mathbb{C}$.

(a) For which matrices A do we have $AM = MA$? Assuming that $AM = MA$, factorise $\det(A)$.
(b) Put $X = \{A \in GL_2\mathbb{C} \mid AM = MA\}$. Show that the map $f\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a + b, a - b)$ gives a homeomorphism $X \to (\mathbb{C} \setminus \{0\}) \times (\mathbb{C} \setminus \{0\})$.
(c) Deduce that X is homotopy equivalent to the torus.

Exercise 7. We write 1 for the space $\{0\}$, with a single point. Explain why $[1, X]$ is essentially the same as $\pi_0(X)$.

Exercise 8. (a) Suppose that X is contractible, so there exists a point $a \in X$ and a map $h\colon I \times X \to X$ such that $h(0, x) = x$ and $h(1, x) = a$ for all $x \in X$. In lectures we gave an indirect proof that X is path-connected. Given points $x, y \in X$, find formulae (in terms of h) for (i) a path from x to a; (ii) a path from y to a; (iii) a path from x to y.

(b) Now suppose we have maps $f\colon X \to Y$ and $g\colon Y \to X$ and a homotopy $k\colon I \times Y \to Y$ between 1_Y and fg. Find a point $b \in Y$ and a contraction of Y to b.

Exercise 9. Find examples of the following things.

(a) A space X, another space $Y \subseteq \mathbb{R}^2$, and two maps $f, g\colon X \to Y$ such that f is homotopic to g, but not linearly homotopic to g.

(b) A connected space X and contractible subspaces $Y, Z \subseteq X$ such that $Y \cap Z$ is connected but not contractible.

(c) Spaces X, Y and an injective map $f\colon X \to Y$ such that $f_*\colon \pi_0 X \to \pi_0 Y$ is not injective.

(d) Spaces X, Y and a surjective map $f\colon X \to Y$ such that $f_*\colon \pi_1 X \to \pi_1 Y$ is not surjective.

Exercise 10. In this problem we study the topology of some spaces occurring in the theory of relativity. Any event in the history of the universe can be located by giving the time t and position (x, y, z) at which it occurs. (Of course, we need to make some agreement about where the origin and the axes are before we can do this.)

Every space described in this problem is either contractible or homotopy equivalent to S^d for some d. When asked to "classify" a space, you should either say that it is contractible, or say what the appropriate d is. Remember that S^0 is the space with two points.

We put

$$\begin{aligned}
L &= \text{ the light cone} \\
&= \{(t, x, y, z) \in \mathbb{R}^4 \mid t^2 - x^2 - y^2 - z^2 = 0\} \\
L^+ &= \text{ the forward light cone} \\
&= \{(t, x, y, z) \in L \mid t > 0\} \\
C &= \text{ the celestial sphere} \\
&= \{(t, x, y, z) \in L^+ \mid t = 1\} \\
H &= \{(t, x, y, z) \in \mathbb{R}^4 \mid t^2 - x^2 - y^2 - z^2 = 1\} \\
H^+ &= \{(t, x, y, z) \in H \mid t > 0\} \\
T &= \{(t, x, y, z) \in \mathbb{R}^4 \mid t^2 - x^2 - y^2 - z^2 = -1\}.
\end{aligned}$$

We refer to all this as the case $n = 3$, because we are using three space coordinates: x, y and z. To help us see what is going on, we will also study the cases $n = 2$ (where we ignore z) and $n = 1$ (where we ignore both y and z). Thus, for example, in the case $n = 1$ we have $L = \{(t, x) \in \mathbb{R}^2 \mid t^2 - x^2 = 0\}$, and in the case $n = 2$ we have $H = \{(t, x, y) \in \mathbb{R}^3 \mid t^2 - x^2 - y^2 = 1\}$.

(a) In the case $n = 1$, draw the subsets L, L^+, C, H, H^+ and T of $\mathbb{R}^2$, and classify them. You should do this "by inspection"; no proof is required.

(b) Now do the same in the case $n = 2$.

(c) In the case $n = 3$, classify the spaces L, L^+, C, H, H^+ and T, giving formulae and proofs.

Exercise 11. Let $f\colon X \to \mathbb{R} \setminus \{0\}$ be a continuous map. Show that f is homotopic to a map g such that $g(x)^2 = 1$ for all $x \in X$.

Exercise 12. Suppose we have maps $W \xrightarrow{f} X \xrightarrow{g} Y$, and that X is contractible.

(a) Prove that f is homotopic to a constant map.

(b) Prove that g is homotopic to a constant map.

Exercise 13. (a) Suppose we have maps $f, g\colon X \to Y$ and $f', g'\colon X' \to Y'$, and that $f \simeq g$ and $f' \simeq g'$. Prove that $f \times f' \simeq g \times g'$ as maps $X \times X' \to Y \times Y'$.

(b) Suppose we have metric spaces X, X', Y, Y' such that X is homotopy equivalent to Y and X' is homotopy equivalent to Y'. Prove that $X \times X'$ is homotopy equivalent to $Y \times Y'$.

Exercise 14. Let f be a polynomial of degree n over $\mathbb{R}$, and suppose that f has n distinct real roots. Put $X = \{x \in \mathbb{R} \mid f(x) \neq 0\}$ and $Y = \{x \in \mathbb{R} \mid f'(x) = 0\}$. How far is X from being homotopy equivalent to Y? What happens if some of the roots of f are not real, or if there are repeated roots?

Exercise 15. Show that the space $X = S^2 \setminus S^1$ of points of the 2-sphere not on the equator is homotopy equivalent to the space $Y = \{N, S\}$ consisting of the north and south poles.