MAS435 ALGEBRAIC TOPOLOGY — PROBLEM SHEET 8

Please hand in Exercises 1 and 2 by the end of Week 11.
Your answers should be submitted via Blackboard. Please either

(a) Use LaTeX, and upload the resulting PDF file; or
(b) Scan handwritten work with a scanning app such as CamScanner (Android/ Apple) or Microsoft Lens (Android/ Apple) or Adobe Scan (Android/ Apple). Again, you should upload files in PDF form.

Exercise 1. Here is an alternative set of formulae that can be used to prove Proposition 11.12:

$$h_0(s,t) = u(\max(2t-1, t+s/2-1/2, 0)) \qquad h_1(s,t) = u(\min(2t, t-s/2+1/2, 1))$$
$$h_2(s,t) = u(\min(2t, 2(1-t), 1-s)) \qquad h_3(a,t) = u(\max(1-2t, 2t-1, s)$$
$$h_4(s,t) = \begin{cases} u(\min(4t, 2t-s/2+1/2)) & \text{if } 4t-s-1 \leq 0 \\ v(4t-s-1) & \text{if } 0 \leq 4t-s-1 \leq 1 \\ w(\max(4t-3, 2t-s/2-1)) & \text{if } 1 \leq 4t-s-1. \end{cases}$$

Spell out all the details for h_0, and give a briefer discussion of the other maps.

Exercise 2. A *topological group* is a set G with both a group structure and a topology, which are compatible. In more detail, the group structure defines maps $\mu\colon G \times G \to G$ and $\chi\colon G \to G$ by $\mu(g,h) = gh$ and $\chi(g) = g^{-1}$; the compatibility condition is that μ and χ should be continuous with respect to the given topology on G and the product topology on $G \times G$.

(a) Explain how S^1 can be considered as a topological group.
(b) Explain how the set $O_n = \{A \in M_n(\mathbb{R}) \mid AA^T = I\}$ can be considered as a topological group.

Now fix a topological group G, and let e be the identity element. Suppose that $u, v\colon e \rightsquigarrow e$ in G. As usual, we define $u * v\colon [0,1] \to G$ to be the joined path. We also define $u \bullet v\colon [0,1] \to G$ by $(u \bullet v)(t) = u(t)v(t)$ (using the group structure of G to multiply the elements $u(t) \in G$ and $v(t) \in G$).

(c) Show that there are path homotopies $u \bullet v \simeq u * v$ and $u \bullet v \simeq v * u$. Deduce that $\pi_1(G, e)$ is abelian.
(d) Can the Klein bottle be made into a topological group?
(e) Show that for any $g \in G$, the group $\pi_1(G, g)$ is isomorphic to $\pi_1(G, e)$ (even if g is not in the same path component as e).

Exercise 3. Recall that the Möbius strip can be defined as

$$M = \{(z, w) \in S^1 \times B^2 \mid w^2/z \in [0,1] \subseteq \mathbb{R} \subseteq \mathbb{C}\}.$$

Show that there is a covering map $p\colon S^1 \times [-1,1] \to M$.