VECTOR SPACES AND FOURIER THEORY

NEIL STRICKLAND

1. Introduction

This course involves many of the same themes as SOM201 (Linear Mathematics for Applications), but takes a more abstract point of view. A central aim of the course is to help you become familiar and comfortable with mathematical abstraction and generalisation, which plays an important role in pure mathematics. This has many benefits. For example, we will be able to prove a single theorem that simultaneously tells us useful things about vectors, matrices, polynomials, differential equations, and sequences satisfying a recurrence relation. Without the axiomatic approach, we would have to give five different (but very similar) proofs, which would be much less efficient. We will also be led to make some non-obvious but useful analogies between different situations. For example, we will be able to define the distance or angle between two functions (by analogy with the distance or angle between two vectors in $\mathbb{R}^3$), and this will help us to understand the theory of Fourier series. We will prove a number of things that were merely stated in SOM201. Similarly, we will give abstract proofs of some things that were previously proved using matrix manipulation. These new proofs will require a better understanding of the underlying concepts, but once you have that understanding, they will often be considerably simpler.

2. Vector spaces

Predefinition 2.1. A vector space (over $\mathbb{R}$) is a nonempty set V of things such that

(a) If u and v are elements of V, then $u + v$ is an also an element of V.
(b) If u is an element of V and t is a real number, then tu is an element of V.

This definition is not strictly meaningful or rigorous; we will pick holes in it later (see Example 2.12). But it will do for the moment.

Example 2.2. The set $\mathbb{R}^3$ of all three-dimensional vectors is a vector space, because the sum of two vectors is a vector (eg $\begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}3\\2\\1\end{bmatrix} = \begin{bmatrix}4\\4\\4\end{bmatrix}$) and the product of a real number and a vector is a vector (eg $3\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}3\\6\\9\end{bmatrix}$). In the same way, the set $\mathbb{R}^2$ of two-dimensional vectors is also a vector space.

Remark 2.3. For various reasons it will be convenient to work mostly with column vectors, as in the previous example. However, this can be typographically awkward, so we use the following notational device: if u is a row vector, then u^T denotes the corresponding column vector, so for example

$$\begin{bmatrix}1 & 2 & 3 & 4\end{bmatrix}^T = \begin{bmatrix}1\\2\\3\\4\end{bmatrix}.$$

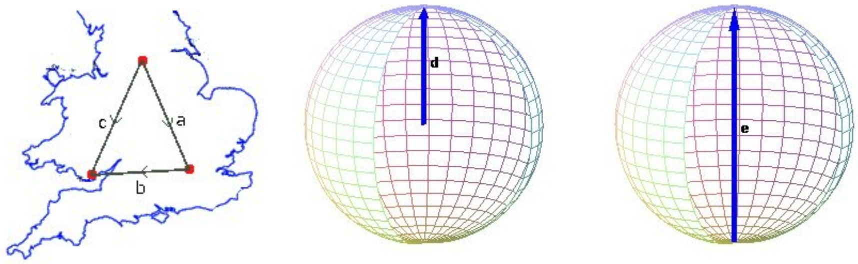
Example 2.4. For any natural number n the set $\mathbb{R}^n$ of vectors of length n is a vector space. For example, the vectors $u = \begin{bmatrix}1 & 2 & 4 & 8 & 16\end{bmatrix}^T$ and $v = \begin{bmatrix}1 & -2 & 4 & -8 & 16\end{bmatrix}^T$ are elements of $\mathbb{R}^5$, with $u + v = \begin{bmatrix}2 & 0 & 8 & 0 & 32\end{bmatrix}^T$. We can even consider the set $\mathbb{R}^\infty$ of all infinite sequences of real numbers, which is again a vector space.

Example 2.5. The set $\{0\}$ is a trivial example of a vector space (but it is important in the same way that the number zero is important). This space can also be thought of as $\mathbb{R}^0$. Another trivial example is that $\mathbb{R}$ itself is a vector space (which can be thought of as $\mathbb{R}^1$).

Example 2.6. The set U of physical vectors is a vector space. We can define some elements of U by

- $\mathbf{a}$ is the vector from Sheffield to London
- $\mathbf{b}$ is the vector from London to Cardiff
- $\mathbf{c}$ is the vector from Sheffield to Cardiff
- $\mathbf{d}$ is the vector from the centre of the earth to the north pole
- $\mathbf{e}$ is the vector from the south pole to the north pole.

We then have $\mathbf{a} + \mathbf{b} = \mathbf{c}$ and $2\mathbf{d} = \mathbf{e}$.



Once we have agreed on where our axes should point, and what units of length we should use, we can identify U with $\mathbb{R}^3$. However, it is conceptually important (especially in the theory of relativity) that U exists in its own right without any such choice of conventions.

Example 2.7. The set $F(\mathbb{R})$ of all functions from $\mathbb{R}$ to $\mathbb{R}$ is a vector space, because we can add any two functions to get a new function, and we can multiply a function by a number to get a new function. For example, we can define functions $f, g, h \colon \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned} f(x) &= e^x \\ g(x) &= e^{-x} \\ h(x) &= \cosh(x) = (e^x + e^{-x})/2, \end{aligned}$$

so f, g and h are elements of $F(\mathbb{R})$. Then $f+g$ and $2h$ are again functions, in other words $f+g \in F(\mathbb{R})$ and $2h \in F(\mathbb{R})$. Of course we actually have $f+g = 2h$ in this example.

For this to work properly, we must insist that $f(x)$ is defined for all x, and is a real number for all x; it cannot be infinite or imaginary. Thus the rules $p(x) = 1/x$ and $q(x) = \sqrt{x}$ do not define elements $p, q \in F(\mathbb{R})$.

Remark 2.8. In order to understand the above example, you need to think of a function $f \colon \mathbb{R} \to \mathbb{R}$ as a single object in its own right, and then think about the set $F(\mathbb{R})$ of all possible functions as a single object; later you will need to think about various different subsets of $F(\mathbb{R})$. All this may seem quite difficult to deal with. However, it is a *central aim* of this course for you to get to grips with this level of abstraction. So you should persevere, ask questions, study the notes and work through examples until it becomes clear to you.

Example 2.9. In practise, we do not generally want to consider the set $F(\mathbb{R})$ of *all* functions. Instead we consider the set $C(\mathbb{R})$ of continuous functions, or the set $C^\infty(\mathbb{R})$ of "smooth" functions (those which can be differentiated arbitrarily often), or the set $\mathbb{R}[x]$ of polynomial functions (eg $p(x) = 1 + x + x^2 + x^3$ defines an element $p \in \mathbb{R}[x]$). If f and g are continuous then $f+g$ and tf are continuous, so $C(\mathbb{R})$ is a vector space. If f and g are smooth then $f+g$ and tf are smooth, so $C^\infty(\mathbb{R})$ is a vector space. If f and g are polynomials then $f+g$ and tf are polynomials, so $\mathbb{R}[x]$ is a vector space.

Example 2.10. We also let $[a, b]$ denote the interval $\{x \in \mathbb{R} \mid a \leq x \leq b\}$, and we write $C[a, b]$ for the set of continuous functions $f \colon [a, b] \to \mathbb{R}$. For example, the rule $f(x) = 1/x$ defines a continuous function on the interval $[1, 2]$. (The only potential problem is at the point $x = 0$, but $0 \notin [1, 2]$, so we do not need to worry about it.) We thus have an element $f \in C[1, 2]$.

Example 2.11. The set $M_2\mathbb{R}$ of 2×2 matrices (with real entries) is a vector space. Indeed, if we add two such matrices, we get another 2×2 matrix, for example

$$\left[\begin{smallmatrix} 1 & 0 \\ 0 & 4 \end{smallmatrix}\right] + \left[\begin{smallmatrix} 0 & 2 \\ 3 & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right].$$

Similarly, if we multiply a 2×2 matrix by a real number, we get another 2×2 matrix, for example

$$7\left[\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 7 & 14 \\ 21 & 28 \end{smallmatrix}\right].$$

We can identify $M_2\mathbb{R}$ with $\mathbb{R}^4$, by the rule

$$\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \leftrightarrow \left[\begin{smallmatrix} a \\ b \\ c \\ d \end{smallmatrix}\right].$$

More generally, for any n the set $M_n\mathbb{R}$ of $n \times n$ square matrices is a vector space, which can be identified with $\mathbb{R}^{n^2}$. More generally still, for any n and m, the set $M_{n,m}\mathbb{R}$ of $n \times m$ matrices is a vector space, which can be identified with $\mathbb{R}^{nm}$.

Example 2.12. Let L be the set of all finite lists of real numbers. For example, the lists $\mathbf{a} = (10, 20, 30, 40)$ and $\mathbf{b} = (5, 6, 7)$ and $\mathbf{c} = (1, \pi, \pi^2)$ define three elements $\mathbf{a}, \mathbf{b}, \mathbf{c} \in L$. Is L a vector space? In trying to answer this question, we will reveal some inadequacies of Predefinition 2.1.

It seems clear that L is closed under scalar multiplication: for example $100\mathbf{b} = (500, 600, 700)$, which is another element of L. The real issue is closure under addition. For example, is $\mathbf{a} + \mathbf{b}$ an element of L? We cannot answer this unless we know what $\mathbf{a} + \mathbf{b}$ means. There are at least three possible meanings:

(1) $\mathbf{a} + \mathbf{b}$ could mean $(10, 20, 30, 40, 5, 6, 7)$ (the list $\mathbf{a}$ followed by the list $\mathbf{c}$).
(2) $\mathbf{a} + \mathbf{b}$ could mean $(15, 26, 37)$ (chop off $\mathbf{a}$ to make the lists the same length, then add them together).
(3) $\mathbf{a} + \mathbf{b}$ could mean $(15, 26, 37, 40)$ (add zeros to the end of $\mathbf{c}$ to make the lists the same length, then add them together.)

The point is that the expression $\mathbf{a} + \mathbf{b}$ does not have a meaning until we decide to give it one. (Strictly speaking, the same is true of the expression $100\mathbf{b}$, but in that case there is only one reasonable possibility for what it should mean.) To avoid this kind of ambiguity, we should say that a vector space is a set *together with a definition of addition* etc.

Suppose we agree to use the third definition of addition, so that $\mathbf{a} + \mathbf{b} = (15, 26, 37, 40)$. The ordinary rules of algebra would tell us that $(\mathbf{a} + (-1).\mathbf{a}) + \mathbf{b} = \mathbf{b}$. However, in fact we have

$$\begin{aligned}(\mathbf{a} + (-1).\mathbf{a}) + \mathbf{b} &= ((10, 20, 30, 40) + (-10, -20, -30, -40)) + (5, 6, 7)\\ &= (0, 0, 0, 0) + (5, 6, 7) = (5, 6, 7, 0)\\ &\neq (5, 6, 7) = \mathbf{b}.\end{aligned}$$

Thus, the ordinary rules of algebra do not hold. We do not want to deal with this kind of thing; we only want to consider sets where addition and scalar multiplication work in the usual way. We must therefore give a more careful definition of a vector space, which will allow us to say that L is not a vector space, so we need not think about it.

(If we used either of the other definitions of addition then things would still go wrong; details are left as an exercise.)

Our next attempt at a definition is as follows:

Predefinition 2.13. A vector space over $\mathbb{R}$ is a nonempty set V, together with a definition of what it means to add elements of V or multiply them by real numbers, such that

(a) If u and v are elements of V, then $u + v$ is an also an element of V.
(b) If u is an element of V and t is a real number, then tu is an element of V.
(c) All the usual algebraic rules for addition and multiplication hold.

In the course we will be content with an informal understanding of the phrase "all the usual algebraic rules", but for completeness, we give an explicit list of axioms:

Definition 2.14. A vector space over $\mathbb{R}$ is a set V, together with an element $0 \in V$ and a definition of what it means to add elements of V or multiply them by real numbers, such that

(a) If u and v are elements of V, then $u + v$ is an also an element of V.
(b) If v is an element of V and t is a real number, then tv is an element of V.
(c) For any elements $u, v, w \in V$ and any real numbers s, t, the following equations hold:
   (1) $0 + v = v$
   (2) $u + v = v + u$
   (3) $u + (v + w) = (u + v) + w$
   (4) $0u = 0$
   (5) $1u = u$
   (6) $(st)u = s(tu)$
   (7) $(s + t)u = su + tu$
   (8) $s(u + v) = su + sv$.

Note that there are many rules that do not appear explicitly in the above list, such as the fact that $t(u+v-w/t) = tu + tv - w$, but it turns out that all such rules can be deduced from the ones listed. We will not discuss any such deductions.

In example 2.12, the only element $0 \in L$ with the property that $0 + v = v$ for all v is the empty list $0 = ()$. If u is a nonempty list of length n, then $0u$ is a list of n zeros, which is not the same as the empty list, so the axiom $0u = 0$ is not satisfied, so L is not a vector space. In all our other examples, it is obvious that the axioms hold, and we will not discuss them further.

Remark 2.15. We will usually use the symbol 0 for the zero element of whatever vector space we are considering. Thus 0 could mean the vector $\left[\begin{smallmatrix}0\\0\\0\end{smallmatrix}\right]$ (if we are working with $\mathbb{R}^3$) or the zero matrix $\left[\begin{smallmatrix}0&0&0\\0&0&0\end{smallmatrix}\right]$ (if we are working with $M_{2,3}\mathbb{R}$) or whatever. Occasionally it will be important to distinguish between the zero elements in different vector spaces. In that case, we write 0_V for the zero element of V. For example, we have $0_{\mathbb{R}^2} = \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$ and $0_{M_2\mathbb{R}} = \left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right]$.

One can also consider vector spaces over fields other than $\mathbb{R}$; the most important case for us will be the field $\mathbb{C}$ of complex numbers. We record the definitions for completeness.

Definition 2.16. A *field* is a set K together with elements $0, 1 \in K$ and a definition of what it means to add or multiply two elements of K, such that:

(a) The usual rules of algebra are valid. More explicitly, for all $a, b, c \in K$ the following equations hold:
- $0 + a = a$
- $a + (b + c) = (a + b) + c$
- $a + b = b + a$
- $1.a = a$
- $a(bc) = (ab)c$
- $ab = ba$
- $a(b + c) = ab + ac$

(b) For every $a \in K$ there is an element $-a$ with $a + (-a) = 0$.
(c) For every $a \in K$ with $a \neq 0$ there is an element $a^{-1} \in K$ with $aa^{-1} = 1$.
(d) $1 \neq 0$.

Example 2.17. Recall that

$$\begin{aligned}
\mathbb{Z} &= \{ \text{ integers } \} = \{\dots, -2, -1, 0, 1, 2, 3, 4, \dots\} \\
\mathbb{Q} &= \{ \text{ rational numbers } \} = \{a/b \mid a, b \in \mathbb{Z}\ ,\ b \neq 0\} \\
\mathbb{R} &= \{ \text{ real numbers } \} \\
\mathbb{C} &= \{ \text{ complex numbers } \} = \{x + iy \mid x, y \in \mathbb{R}\},
\end{aligned}$$

so $\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$. Then $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Q}$ are fields. The ring $\mathbb{Z}$ is not a field, because axiom (c) is not satisfied: there is no element 2^{-1} *in the set* $\mathbb{Z}$ for which $2.2^{-1} = 1$. One can show that the ring $\mathbb{Z}/n\mathbb{Z}$ is a field if and only if n is a prime number.

Definition 2.18. A vector space over a field K is a set V, together with an element $0 \in V$ and a definition of what it means to add elements of V or multiply them by elements of K, such that

(a) If u and v are elements of V, then $u + v$ is an also an element of V.
(b) If v is an element of V and t is an element of K, then tv is an element of V.
(c) For any elements $u, v, w \in V$ and any elements $s, t \in K$, the following equations hold:
  (1) $0 + v = v$
  (2) $u + v = v + u$
  (3) $u + (v + w) = (u + v) + w$
  (4) $0u = 0$
  (5) $1u = u$
  (6) $(st)u = s(tu)$
  (7) $(s + t)u = su + tu$
  (8) $s(u + v) = su + sv$.

Example 2.19. Almost all our examples of real vector spaces work over any field K. For example, the set $M_4\mathbb{Q}$ (of 4×4 matrices whose entries are rational numbers) is a vector space over $\mathbb{Q}$. The set $\mathbb{C}[x]$ (of polynomials with complex coefficients) is a vector space over $\mathbb{C}$.

3. Linear maps

Definition 3.1. Let V and W be vector spaces, and let $\phi\colon V \to W$ be a function (so for each element $v \in V$ we have an element $\phi(v) \in W$). We say that ϕ is *linear* if

(a) For any v and v' in V, we have $\phi(v + v') = \phi(v) + \phi(v')$ in W.
(b) For any $t \in \mathbb{R}$ and $v \in V$ we have $\phi(tv) = t\phi(v)$ in W.

By taking $t = v = 0$ in (b), we see that a linear map must satisfy $\phi(0) = 0$. Further simple arguments also show that $\phi(v - v') = \phi(v) - \phi(v')$.

Remark 3.2. The definition can be reformulated slightly as follows. A map $\phi\colon V \to W$ is linear iff

(c) For any $t, t' \in \mathbb{R}$ and any $v, v' \in V$ we have $\phi(tv + t'v') = t\phi(v) + t'\phi(v')$.

To show that this reformulation is valid, we must show that if (c) holds, then so do (a) and (b); and conversely, if (a) and (b) hold, then so does (c).

Condition (a) is the special case of (c) where $t = t' = 1$, and condition (b) is the special case where $t' = 0$ and $v' = 0$. Thus, if (c) holds then so do (a) and (b). Conversely, suppose that (a) and (b) hold, and that we have $t, t' \in \mathbb{R}$

and $v, v' \in V$. Condition (a) tells us that $\phi(tv + t'v') = \phi(tv) + \phi(t'v')$, and condition (b) tells us that $\phi(tv) = t\phi(v)$ and $\phi(t'v') = t'\phi(v')$. Putting these together, we get

$$\phi(tv + t'v') = t\phi(v) + t'\phi(v'),$$

so condition (c) holds, as required.

Example 3.3. Consider the functions $f, g\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2x$ and $g(x) = x^2$. Then

$$g(x + x') = x^2 + x'^2 + 2xx' \neq x^2 + x'^2 = g(x) + g(x'),$$

so g is not linear. Similarly, for general x and x' we have $\sin(x+x') \neq \sin(x)+\sin(x')$ and $\exp(x+x') \neq \exp(x)+\exp(x')$, so the functions sin and exp are not linear. On the other hand, we have

$$\begin{aligned} f(x + x') &= 2(x + x') = 2x + 2x' = f(x) + f(x') \\ f(tx) &= 2tx = tf(x) \end{aligned}$$

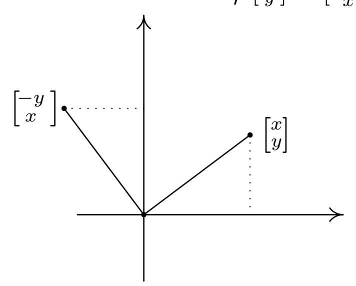
so f is linear.

Example 3.4. The obvious generalisation of the previous example is as follows. For any number $m \in \mathbb{R}$, we can define $\mu_m\colon \mathbb{R} \to \mathbb{R}$ by $\mu_m(x) = mx$ (so f in the previous example is μ_2). We have

$$\begin{aligned} \mu_m(x + x') &= m(x + x') = mx + mx' = \mu_m(x) + \mu_m(x') \\ \mu_m(tx) &= mtx = tmx = t\mu_m(x), \end{aligned}$$

so μ_m is linear (and in fact, these are all the linear maps from $\mathbb{R}$ to $\mathbb{R}$). Note also that the graph of μ_m is a straight line of slope m through the origin; this is essentially the reason for the word "linear".

Example 3.5. For any $\mathbf{v} \in \mathbb{R}^2$, we let $\rho(\mathbf{v})$ be the vector obtained by rotating $\mathbf{v}$ through 90 degrees anticlockwise around the origin. It is well-known that the formula for this is $\rho\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right] = \left[\begin{smallmatrix}-y\\x\end{smallmatrix}\right]$.

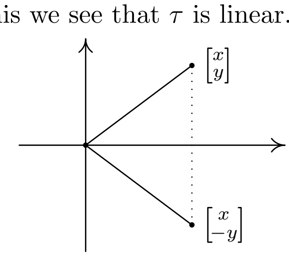


We thus have

$$\begin{aligned} \rho\left(\begin{bmatrix}x\\y\end{bmatrix} + \begin{bmatrix}x'\\y'\end{bmatrix}\right) &= \rho\begin{bmatrix}x+x'\\y+y'\end{bmatrix} = \begin{bmatrix}-y-y'\\x+x'\end{bmatrix} = \begin{bmatrix}-y\\x\end{bmatrix} + \begin{bmatrix}-y'\\x'\end{bmatrix} = \rho\begin{bmatrix}x\\y\end{bmatrix} + \rho\begin{bmatrix}x'\\y'\end{bmatrix} \\ \rho\left(t\begin{bmatrix}x\\y\end{bmatrix}\right) &= \rho\begin{bmatrix}tx\\ty\end{bmatrix} = \begin{bmatrix}-ty\\tx\end{bmatrix} = t\rho\begin{bmatrix}x\\y\end{bmatrix}, \end{aligned}$$

so ρ is linear. (Can you explain this geometrically, without using the formula?)

Example 3.6. For any $\mathbf{v} \in \mathbb{R}^2$, we let $\tau(\mathbf{v})$ be the vector obtained by reflecting $\mathbf{v}$ across the line $y = 0$. It is clear that the formula is $\tau\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right] = \left[\begin{smallmatrix}x\\-y\end{smallmatrix}\right]$, and using this we see that τ is linear.



Example 3.7. Define $\theta\colon \mathbb{R}^2 \to \mathbb{R}$ by $\theta(\mathbf{v}) = \|\mathbf{v}\|$, so $\theta\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right] = \sqrt{x^2 + y^2}$. This is not linear, because $\theta(\mathbf{u} + \mathbf{v}) \neq \theta(\mathbf{u}) + \theta(\mathbf{v})$ in general. Indeed, if $\mathbf{u} = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $\mathbf{v} = \left[\begin{smallmatrix}-1\\0\end{smallmatrix}\right]$ then $\theta(\mathbf{u} + \mathbf{v}) = 0$ but $\theta(\mathbf{u}) + \theta(\mathbf{v}) = 1 + 1 = 2$.

Example 3.8. Define $\sigma\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $\sigma\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right] = \left[\begin{smallmatrix}x+1\\y-1\end{smallmatrix}\right]$. Then σ is not linear, because $\sigma\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] \neq \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$.

Example 3.9. Define $\alpha\colon \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\alpha\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right] = \begin{bmatrix}y^3/(x^2+y^2)\\x^3/(x^2+y^2)\end{bmatrix}.$$

(This does not really make sense when $x = y = 0$, but for that case we make the separate definition that $\alpha\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] = \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$.) This map satisfies $\alpha(t\mathbf{v}) = t\alpha(\mathbf{v})$, but it does not satisfy $\alpha(\mathbf{u} + \mathbf{v}) = \alpha(\mathbf{u}) + \alpha(\mathbf{v})$, so it is not linear. For example, if $\mathbf{u} = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $\mathbf{v} = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ then $\alpha(\mathbf{u}) = \mathbf{v}$ and $\alpha(\mathbf{v}) = \mathbf{u}$ but $\alpha(\mathbf{u} + \mathbf{v}) = (\mathbf{u} + \mathbf{v})/2 \neq \alpha(\mathbf{u}) + \alpha(\mathbf{v})$.

Example 3.10. Given vectors $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ in $\mathbb{R}^3$, recall that the inner product and cross product are defined by

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}.\mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$
$$\mathbf{u} \times \mathbf{v} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}.$$

Fix a vector $\mathbf{a} \in \mathbb{R}^3$. Define $\alpha \colon \mathbb{R}^3 \to \mathbb{R}$ by $\alpha(\mathbf{v}) = \langle \mathbf{a}, \mathbf{v} \rangle$ and $\beta \colon \mathbb{R}^3 \to \mathbb{R}^3$ by $\beta(\mathbf{v}) = \mathbf{a} \times \mathbf{v}$. Then both α and β are linear. To prove this we must show that $\alpha(t\mathbf{v}) = t\alpha(\mathbf{v})$ and $\alpha(\mathbf{v} + \mathbf{w}) = \alpha(\mathbf{v}) + \alpha(\mathbf{w})$ and $\beta(t\mathbf{v}) = t\beta(\mathbf{v})$ and $\beta(\mathbf{v} + \mathbf{w}) = \beta(\mathbf{v}) + \beta(\mathbf{w})$. We will write out only the last of these; the others are similar but easier.

$$\beta(\mathbf{v} + \mathbf{w}) = \beta \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 \end{bmatrix} = \begin{bmatrix} a_2(v_3 + w_3) - a_3(v_2 + w_2) \\ a_3(v_1 + w_1) - a_1(v_3 + w_3) \\ a_1(v_2 + w_2) - a_2(v_1 + w_1) \end{bmatrix} = \begin{bmatrix} a_2 v_3 - a_3 v_2 \\ a_3 v_1 - a_1 v_3 \\ a_1 v_2 - a_2 v_1 \end{bmatrix} + \begin{bmatrix} a_2 w_3 - a_3 w_2 \\ a_3 w_1 - a_1 w_3 \\ a_1 w_2 - a_2 w_1 \end{bmatrix} = \beta(\mathbf{v}) + \beta(\mathbf{w}).$$

Example 3.11. Let A be a fixed $m \times n$ matrix. Given a vector $\mathbf{v}$ of length n (so $\mathbf{v} \in \mathbb{R}^n$), we can multiply A by $\mathbf{v}$ in the usual way to get a vector $A\mathbf{v}$ of length m. We can thus define $\phi_A \colon \mathbb{R}^n \to \mathbb{R}^m$ by $\phi_A(\mathbf{v}) = A\mathbf{v}$. It is clear that $A(\mathbf{v} + \mathbf{v}') = A\mathbf{v} + A\mathbf{v}'$ and $At\mathbf{v} = tA\mathbf{v}$, so ϕ_A is a linear map. We will see later that every linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$ has this form. In particular, if we put

$$R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

we find that

$$R \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix} = \rho\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) \qquad T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix} = \tau\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$$

(where ρ and τ are as in Examples 3.5 and 3.6). This means that $\rho = \phi_R$ and $\tau = \phi_T$.

Example 3.12. For any continuous function $f \colon \mathbb{R} \to \mathbb{R}$, we write

$$I(f) = \int_0^1 f(x)dx \in \mathbb{R}.$$

This defines a map $I \colon C(\mathbb{R}) \to \mathbb{R}$. If we put

$$p(x) = x^2$$
$$q(x) = 2x - 1$$
$$r(x) = e^x$$

we have $I(p) = 1/3$ and $I(q) = 0$ and $I(r) = e - 1$.

Using the obvious equations

$$\int_0^1 f(x) + g(x)dx = \int_0^1 f(x)dx + \int_0^1 g(x)dx$$
$$\int_0^1 tf(x)dx = t \int_0^1 f(x)dx$$

we see that I is a linear map.

Definition 3.13. For any smooth function $f \colon \mathbb{R} \to \mathbb{R}$ we write $D(f) = f'$ and $L(f) = f'' + f$. These are again smooth functions, so we have maps $D \colon C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ and $L \colon C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$. If we put

$$p(x) = \sin(x)$$
$$q(x) = \cos(x)$$
$$r(x) = e^x$$

then $D(p) = q$ and $D(q) = -p$ and $D(r) = r$. It follows that $L(p) = L(q) = 0$ and that $L(r) = 2r$. Using the obvious equations

$$(f + g)' = f' + g'$$
$$(tf)' = t\, f'$$

we see that D is linear. Similarly, we have

$$\begin{aligned} L(f + g) &= (f + g)'' + (f + g) = f'' + g'' + f + g \\ &= (f'' + f) + (g'' + g) = L(f) + L(g) \\ L(tf) &= (tf)'' + tf = t\, f'' + tf \\ &= tL(f). \end{aligned}$$

This shows that L is also linear.

Example 3.14. For any 2×2 matrix $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, the trace and determinant are defined by $\text{trace}(A) = a + d \in \mathbb{R}$ and $\det(A) = ad - bc \in \mathbb{R}$. We thus have two functions trace, det: $M_2\mathbb{R} \to \mathbb{R}$. It is easy to see that $\text{trace}(A + B) = \text{trace}(A) + \text{trace}(B)$ and $\text{trace}(tA) = t\,\text{trace}(A)$, so trace: $M_2\mathbb{R} \to \mathbb{R}$ is a linear map. On the other hand, we have $\det(tA) = t^2 \det(A)$ and $\det(A+B) \neq \det(A)+\det(B)$ in general, so det: $M_2\mathbb{R} \to \mathbb{R}$ is not a linear map. For a specific counterexample, consider

$$A = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right] \qquad \text{and} \qquad B = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$$

Then $\det(A) = \det(B) = 0$ but $\det(A + B) = 1$, so $\det(A + B) \neq \det(A) + \det(B)$.

None of this is really restricted to 2×2 matrices. For any n we have a map trace: $M_n\mathbb{R} \to \mathbb{R}$ given by $\text{trace}(A) = \sum_{i=1}^{n} A_{ii}$, which is again linear. We also have a determinant map det: $M_n\mathbb{R} \to \mathbb{R}$ which satisfies $\det(tI) = t^n$; this shows that det is not linear, except in the silly case where $n = 1$.

Example 3.15. "Define" $\phi\colon M_2\mathbb{R} \to M_2\mathbb{R}$ by $\phi(A) = A^{-1}$, so

$$\phi \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = \left[\begin{smallmatrix} d/(ad-bc) & -b/(ad-bc) \\ -c/(ad-bc) & a/(ad-bc) \end{smallmatrix}\right].$$

This is not a linear map, simply because it is not a well-defined map at all: the "definition" does not make sense when $ad - bc = 0$. Even if it were well-defined, it would not be linear, because $\phi(I + I) = (2I)^{-1} = I/2$, whereas $\phi(I) + \phi(I) = 2I$, so $\phi(I + I) \neq \phi(I) + \phi(I)$.

Example 3.16. Define $\phi\colon M_3\mathbb{R} \to M_3\mathbb{R}$ by

$$\phi(A) = \text{ the row reduced echelon form of } A.$$

For example, we have the following sequence of reductions:

$$\left[\begin{smallmatrix} 1 & 2 & 3 \\ 4 & 8 & 6 \\ 7 & 14 & 9 \end{smallmatrix}\right] \to \left[\begin{smallmatrix} 1 & 2 & 3 \\ 0 & 0 & -6 \\ 0 & 0 & -12 \end{smallmatrix}\right] \to \left[\begin{smallmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & -12 \end{smallmatrix}\right] \to \left[\begin{smallmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right],$$

which shows that

$$\phi \left[\begin{smallmatrix} 1 & 2 & 3 \\ 4 & 8 & 6 \\ 7 & 14 & 9 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right].$$

The map is not linear, because $\phi(I) = I$ and also $\phi(2I) = I$, so $\phi(2I) \neq 2\phi(I)$.

Example 3.17. We can define a map trans: $M_n\mathbb{R} \to M_n\mathbb{R}$ by $\text{trans}(A) = A^T$. Here as usual, A^T is the transpose of A, which is obtained by flipping A across the main diagonal. For example:

$$\left[\begin{smallmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{smallmatrix}\right]^T = \left[\begin{smallmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{smallmatrix}\right].$$

In general, we have $(A^T)_{ij} = A_{ji}$. It is clear that $(A + B)^T = A^T + B^T$ and $(tA)^T = tA^T$, so trans: $M_n\mathbb{R} \to M_n\mathbb{R}$ is a linear map.

Definition 3.18. We say that a linear map $\phi\colon V \to W$ is an *isomorphism* if it is a bijection, so there is an inverse map $\phi^{-1}\colon W \to V$ with $\phi(\phi^{-1}(w)) = w$ for all $w \in W$, and $\phi^{-1}(\phi(v)) = v$ for all $v \in V$. (It turns out that ϕ^{-1} is automatically a *linear* map - we leave this as an exercise.) We say that V and W are *isomorphic* if there exists an isomorphism from V to W.

Example 3.19. We can now rephrase part of Example 2.11 as follows: there is an isomorphism $\phi\colon M_2\mathbb{R} \to \mathbb{R}^4$ given by $\phi \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = [a, b, c, d]^T$, so $M_2\mathbb{R}$ is isomorphic to $\mathbb{R}^4$. Similarly, the space $M_{p,q}\mathbb{R}$ is isomorphic to $\mathbb{R}^{pq}$.

Example 3.20. Let U be the space of physical vectors, as in Example 2.6. A choice of axes and length units gives rise to an isomorphism from $\mathbb{R}^3$ to U. More explicitly, choose a point P on the surface of the earth (for example, the base of the Eiffel Tower) and put

$$\begin{aligned} \mathbf{u} &= \text{ the vector of length 1 km pointing east from } P \\ \mathbf{v} &= \text{ the vector of length 1 km pointing north from } P \\ \mathbf{w} &= \text{ the vector of length 1 km pointing vertically upwards from } P. \end{aligned}$$

Define $\phi\colon \mathbb{R}^3 \to U$ by $\phi(x, y, z) = x\mathbf{u} + y\mathbf{v} + z\mathbf{w}$. Then ϕ is an isomorphism.

We will be able to give more interesting examples of isomorphisms after we have learnt about subspaces.

4. Subspaces

Definition 4.1. Let V be a vector space. A *vector subspace* (or just *subspace*) of V is a subset $W \subseteq V$ such that

(a) $0 \in W$
(b) Whenever u and v lie in W, the element $u+v$ also lies in W. (In other words, W is closed under addition.)
(c) Whenever u lies in W and t lies in $\mathbb{R}$, the element tu also lies in W. (In other words, W is closed under scalar multiplication.)

These conditions mean that W is itself a vector space.

Remark 4.2. Strictly speaking, a vector space is a set *together with a definition of addition and scalar multiplication* such that certain identities hold. We should therefore specify that addition in W is to be defined using the same rule as for V, and similarly for scalar multiplication.

Remark 4.3. The definition can be reformulated slightly as follows: a set $W \subseteq V$ is a subspace iff

(a) $0 \in W$
(d) Whenever $u, v \in W$ and $t, s \in \mathbb{R}$ we have $tu + sv \in W$.

To show that this reformulation is valid, we must check that if condition (d) holds then so do (b) and (c); and conversely, that if (b) and (c) hold then so does (d).

In fact, conditions (b) is the special cases of (d) where $t = s = 1$, and condition (c) is the special case of (d) where $v = 0$; so if (d) holds then so do (b) and (c). Conversely, suppose that (b) and (c) hold, and that $u, v \in W$ and $t, s \in \mathbb{R}$. Then condition (c) tells us that $tu \in W$, and similarly that $sv \in W$. Given these, condition (b) tells us that $tu + sv \in W$; we conclude that condition (d) holds, as required.

Example 4.4. There are two silly examples: $\{0\}$ is always a subspace of V, and V itself is always a subspace of V.

Example 4.5. Any straight line through the origin is a subspace of $\mathbb{R}^2$. These are the only subspaces of $\mathbb{R}^2$ (except for the two silly examples).

Example 4.6. In $\mathbb{R}^3$, any straight line through the origin is a subspace, and any plane through the origin is also a subspace. These are the only subspaces of $\mathbb{R}^3$ (except for the two silly examples).

Example 4.7. The set $W = \{A \in M_2\mathbb{R} \mid \operatorname{trace}(A) = 0\}$ is a subspace of $M_2\mathbb{R}$. To check this, we first note that $0 \in W$. Suppose that $A, A' \in W$ and $t, t' \in \mathbb{R}$. We then have $\operatorname{trace}(A) = \operatorname{trace}(A') = 0$ (because $A, A' \in W$) and so

$$\operatorname{trace}(tA + t'A') = t\operatorname{trace}(A) + t'\operatorname{trace}(A') = t.0 + t'.0 = 0,$$

so $tA + t'A' \in W$. Thus, conditions (a) and (d) in Remark 4.3 is satisfied, showing that W is a subspace as claimed.

Example 4.8. Recall that $\mathbb{R}[x]$ denotes the set of all polynomial functions in one variable (so the functions $p(x) = x+1$ and $q(x) = (x+1)^5 - (x-1)^5$ and $r(x) = 1 + 4x^4 + 8x^8$ define elements $p, q, r \in \mathbb{R}[x]$). It is clear that the sum of two polynomials is another polynomial, and any polynomial multiplied by a constant is also a polynomial, so $\mathbb{R}[x]$ is a subspace of the vector space $F(\mathbb{R})$ of all functions on $\mathbb{R}$.

We write $\mathbb{R}[x]_{\leq d}$ for the set of polynomials of degree at most d, so a general element $f \in \mathbb{R}[x]_{\leq d}$ has the form

$$f(x) = a_0 + a_1 x + \ldots + a_d x^d = \sum_{i=0}^{d} a_i x^i$$

for some $a_0, \ldots, a_d \in \mathbb{R}$. It is easy to see that this is a subspace of $\mathbb{R}[x]$.

If we let f correspond to the vector $\left[\begin{smallmatrix} a_0 & \cdots & a_d \end{smallmatrix}\right]^T \in \mathbb{R}^{d+1}$, we get a one-to-one correspondence between $\mathbb{R}[x]_{\leq d}$ and $\mathbb{R}^{d+1}$. More precisely, there is an isomorphism $\phi\colon \mathbb{R}^{d+1} \to \mathbb{R}[x]_{\leq d}$ given by

$$\phi(\left[\begin{smallmatrix} a_0 & \cdots & a_d \end{smallmatrix}\right]^T) = \sum_{i=0}^{d} a_i x^i.$$

Remark 4.9. It is a common mistake to think that $\mathbb{R}[x]_{\leq d}$ is isomorphic to $\mathbb{R}^d$ (rather than $\mathbb{R}^{d+1}$), but this is not correct. Note that the list $0, 1, 2, 3$ has four entries (not three), and similarly, the list $0, 1, 2, \ldots, d$ has $d+1$ entries (not d).

Example 4.10. A function $f\colon \mathbb{R} \to \mathbb{R}$ is said to be *even* if $f(-x) = f(x)$ for all x, and *odd* if $f(-x) = -f(x)$ for all x. For example, $\cos(-x) = \cos(x)$ and $\sin(-x) = -\sin(x)$, so cos is even and sin is odd. (Of course, most functions are neither even nor odd.) We write EF for the set of even functions, so EF is a subset of the set $F(\mathbb{R})$ of all functions from $\mathbb{R}$ to $\mathbb{R}$, and $\cos \in EF$. If f and g are even, it is clear that $f + g$ is also even. If f is even and t is a constant, then it is clear that tf is also even; and the zero function is certainly even as well. This shows that EF is actually a subspace of $F(\mathbb{R})$. Similarly, the set OF of odd functions is a subspace of $F(\mathbb{R})$.

Example 4.11. Let V be the vector space of smooth functions $u(x,t)$ in two variables x and t (to be thought of as position and time).

- We say that u is a solution of the Wave Equation if
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} = 0.$$
This equation governs the propagation of small waves in deep water, or of electromagnetic waves in empty space.
- We say that u is a solution of the Heat Equation if
$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0.$$
This governs the flow of heat along an iron bar.
- We say that u is a solution of the Korteweg-de Vries (KdV) equation if
$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} - 6u\frac{\partial u}{\partial x} = 0.$$
This equation governs the propagation of large waves in shallow water.

The set of solutions of the Wave Equation is a vector subspace of V, as is the set of solutions to the Heat Equation. However, the sum of two solutions to the KdV equation does not satisfy the KdV equation, so the set of solutions is not a subspace of V. In other words, the Wave Equation and the Heat Equation are linear, but the KdV equation is not.

The distinction between linear and nonlinear differential equations is of fundamental importance in physics. Linear equations can generally be solved analytically, or by efficient computer algorithms, but nonlinear equations require far more computing power. The equations of electromagnetism are linear, which explains why hundreds of different radio, TV and mobile phone channels can coexist, together with visible light (which is also a form of electromagnetic radiation), with little or no interference. The motion of fluids and gasses is governed by the Navier-Stokes equation, which is nonlinear; because of this, massive supercomputers are needed for weather forecasting, climate modelling, and aircraft design.

Example 4.12. Consider the following sets of 3×3 matrices:

$$\begin{aligned}
U_0 &= \{\text{symmetric matrices}\} && = \{A \in M_3\mathbb{R} \mid A^T = A\} \\
U_1 &= \{\text{antisymmetric matrices}\} && = \{A \in M_3\mathbb{R} \mid A^T = -A\} \\
U_2 &= \{\text{trace-free matrices}\} && = \{A \in M_3\mathbb{R} \mid \text{trace}(A) = 0\} \\
U_3 &= \{\text{ diagonal matrices }\} && = \{A \in M_3\mathbb{R} \mid A_{ij} = 0 \text{ whenever } i \neq j\} \\
U_4 &= \{\text{ strictly upper-triangular matrices }\} && = \{A \in M_3\mathbb{R} \mid A_{ij} = 0 \text{ whenever } i \geq j\} \\
U_5 &= \{\text{ invertible matrices }\} && = \{A \in M_3\mathbb{R} \mid \det(A) \neq 0\} \\
U_6 &= \{\text{ noninvertible matrices }\} && = \{A \in M_3\mathbb{R} \mid \det(A) = 0\}
\end{aligned}$$

Then $U_0, \dots, U_4$ are all subspaces of $M_3\mathbb{R}$. We will prove this for U_0 and U_4; the other cases are similar. Firstly, it is clear that the zero matrix has $0^T = 0$, so $0 \in U_0$. Suppose that $A, B \in U_0$ (so $A^T = A$ and $B^T = B$) and $s, t \in \mathbb{R}$. Then
$$(sA + tB)^T = sA^T + tB^T = sA + tB,$$
so $sA + tB \in U_0$. Using Remark 4.3, we conclude that U_0 is a subspace. Now consider U_4. The elements of U_4 are the matrices of the form
$$A = \begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{bmatrix}$$
In particular, the zero matrix is an element of U_4 (with $a_{12} = a_{13} = a_{23} = 0$). Now suppose that $A, B \in U_4$ and $s, t \in \mathbb{R}$. We have
$$sA + tB = s\begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{bmatrix} + t\begin{bmatrix} 0 & b_{12} & b_{13} \\ 0 & 0 & b_{23} \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & sa_{12} + tb_{12} & sa_{13} + tb_{13} \\ 0 & 0 & sa_{23} + tb_{23} \\ 0 & 0 & 0 \end{bmatrix},$$
which shows that $sA + tB$ is again strictly upper triangular, and so is an element of U_4. Using Remark 4.3 again, we conclude that U_4 is also a subspace.

On the other hand, U_5 is not a subspace, because it does not contain the zero matrix. Similarly, U_6 is not a subspace: if we put

$$A = \begin{bmatrix} 1&0&0 \\ 0&0&0 \\ 0&0&0 \end{bmatrix} \qquad\qquad B = \begin{bmatrix} 0&0&0 \\ 0&1&0 \\ 0&0&1 \end{bmatrix}$$

then $A, B \in U_6$ but $A + B = I \notin U_6$.

Definition 4.13. Let U be a vector space, and let V and W be subspaces of U. We put

$$V + W = \{u \in U \mid u = v + w \text{ for some } v \in V \text{ and } w \in W\}.$$

Example 4.14. If $U = \mathbb{R}^3$ and

$$V = \{\begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} \mid x \in \mathbb{R}\} \qquad\qquad W = \{\begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} \mid z \in \mathbb{R}\}$$

then

$$V + W = \{\begin{bmatrix} x \\ 0 \\ z \end{bmatrix} \mid x, z \in \mathbb{R}\}$$

Example 4.15. If $U = M_2\mathbb{R}$ and

$$V = \{\left[\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right] \mid a, b \in \mathbb{R}\} \qquad\qquad W = \{\left[\begin{smallmatrix} 0 & b \\ 0 & d \end{smallmatrix}\right] \mid b, d \in \mathbb{R}\}$$

then

$$V + W = \{\left[\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right] \mid a, b, d \in \mathbb{R}\}.$$

Indeed, any matrix of the form $A = \left[\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right]$ can be written as $A = \left[\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right] + \left[\begin{smallmatrix} 0 & 0 \\ 0 & d \end{smallmatrix}\right]$ with $\left[\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right] \in V$ and $\left[\begin{smallmatrix} 0 & 0 \\ 0 & d \end{smallmatrix}\right] \in W$, so $A \in V + W$. Conversely, any $A \in V + W$ can be written as $A = B + C$ with $B \in V$ and $C \in W$. This means that B has the form $B = \left[\begin{smallmatrix} a & b_1 \\ 0 & 0 \end{smallmatrix}\right]$ for some $a, b_1 \in \mathbb{R}$ and C has the form $C = \left[\begin{smallmatrix} 0 & b_2 \\ 0 & d \end{smallmatrix}\right]$ for some $b_2, d \in \mathbb{R}$, so $A = \left[\begin{smallmatrix} a & b_1+b_2 \\ 0 & d \end{smallmatrix}\right]$, which lies in $V + W$.

Proposition 4.16. *Let U be a vector space, and let V and W be subspaces of U. Then both $V \cap W$ and $V + W$ are subspaces of U.*

Proof. We first consider $V \cap W$. As V is a subspace we have $0 \in V$, and as W is a subspace we have $0 \in W$, so $0 \in V \cap W$. Next, suppose we have $x, x' \in V \cap W$. Then $x, x' \in V$ and V is a subspace, so $x + x' \in V$. Similarly, we have $x, x' \in W$ and W is a subspace so $x + x' \in W$. This shows that $x + x' \in V \cap W$, so $V \cap W$ is closed under addition. Finally consider $x \in V \cap W$ and $t \in \mathbb{R}$. Then $x \in V$ and V is a subspace so $tx \in V$. Similarly $x \in W$ and W is a subspace so $tx \in W$. This shows that $tx \in V \cap W$, so $V \cap W$ is closed under scalar multiplication, so $V \cap W$ is a subspace.

Now consider the space $V + W$. We can write 0 as $0 + 0$ with $0 \in V$ and $0 \in W$, so $0 \in V + W$. Now suppose we have $x, x' \in V + W$. As $x \in V + W$ we can find $v \in V$ and $w \in W$ such that $x = v + w$. As $x' \in V + W$ we can also find $v' \in V$ and $w' \in W$ such that $x' = v' + w'$. We then have $v + v' \in V$ (because V is closed under addition) and $w + w' \in W$ (because W is closed under addition). We also have $x + x' = (v + v') + (w + w')$ with $v + v' \in V$ and $w + w' \in W$, so $x + x' \in V + W$. This shows that $V + W$ is closed under addition. Now suppose we have $t \in \mathbb{R}$. Then $tv \in V$ (because V is closed under scalar multiplication) and $tw \in W$ (because W is closed under scalar multiplication). We thus have $tx = tv + tw$ with $tv \in V$ and $tw \in W$, so $tx \in V + W$. This shows that $V + W$ is also closed under scalar multiplication, so it is a subspace. □

Example 4.17. Take $U = \mathbb{R}^3$ and

$$V = \{[x, y, z]^T \mid x + 2y + 3z = 0\}$$
$$W = \{[x, y, z]^T \mid 3x + 2y + z = 0\}.$$

We claim that

$$V \cap W = \{[x, -2x, x]^T \mid x \in \mathbb{R}\}.$$

and $V + W = \mathbb{R}^3$. Indeed, a vector $[x, y, z]^T$ lies in $V \cap W$ iff we have $x + 2y + 3z = 0$ and also $3x + 2y + z = 0$. If we subtract these two equations and divide by two, we find that $z = x$. If we feed this back into the first equation, we see that $y = -2x$. Conversely, if $y = -2x$ and $z = x$ we see directly that both equations are satisfied. It follows that $V \cap W = \{[x, -2x, x]^T \mid t \in \mathbb{R}\}$ as claimed.

Next, consider an arbitrary vector $\mathbf{u} = [x, y, z] \in \mathbb{R}^3$. Put

$$\mathbf{v} = \frac{1}{12}\begin{bmatrix} 12x + 8y + 4z \\ 3x + 2y + z \\ -6x - 4y - 2z \end{bmatrix} \qquad\qquad \mathbf{w} = \frac{1}{12}\begin{bmatrix} -8y - 4z \\ -3x + 10y - z \\ 6x + 4y + 14z \end{bmatrix},$$

so $\mathbf{v}+\mathbf{w}=\mathbf{u}$. One can check that

$$\begin{aligned}(12x+8y+4z)+2(3x+2y+z)+3(-6x-4y-2z)&=0\\ 3(-8y-4z)+2(-3x+10y-z)+(6x+4y+14z)&=0\end{aligned}$$

so $\mathbf{v}\in V$ and $\mathbf{w}\in W$. This shows that $\mathbf{u}\in V+W$, so $V+W=\mathbb{R}^3$.

Example 4.18. Take

$$\begin{aligned}U&=\mathbb{R}[x]_{\leq 4}\\ V&=\{f\in U\mid f(0)=f'(0)=0\}\\ W&=\{f\in U\mid f(-x)=f(x)\text{ for all }x\}.\end{aligned}$$

To understand these, it is best to write the defining conditions more explicitly in terms of the coefficients of f. Any element $f\in U$ can be written as $f(x)=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4$ for some $a_0,\dots,a_4$. We then have

$$\begin{aligned}f(0)&=a_0\\ f'(x)&=a_1+2a_2x+3a_3x^2+4a_4x^3\\ f'(0)&=a_1\\ f(-x)&=a_0-a_1x+a_2x^2-a_3x^3+a_4x^4\\ f(x)-f(-x)&=2a_1x+2a_3x^3\end{aligned}$$

Thus $f\in V$ iff $a_0=a_1=0$, and $f\in W$ iff $f(x)=f(-x)$ iff $a_1=a_3=0$. This means that

$$\begin{aligned}U&=\{a_0+a_1x+a_2x^2+a_3x^3+a_4x^4\mid a_0,\dots,a_4\in\mathbb{R}\}\\ V&=\{a_2x^2+a_3x^3+a_4x^4\mid a_2,a_3,a_4\in\mathbb{R}\}\\ W&=\{a_0+a_2x^2+a_4x^4\mid a_0,a_2,a_4\in\mathbb{R}\}\end{aligned}$$

From this we see that $f\in V\cap W$ iff $a_0=a_1=a_3=0$, so

$$V\cap W=\{a_2x^2+a_4x^4\mid a_2,a_4\in\mathbb{R}\}.$$

We next claim that $f\in V+W$ iff $a_1=0$, so f has no term in x^1. Indeed, from the formulae above we see that any polynomial in V or in W has no term in x^1, so if we add together a polynomial in V and a polynomial in W we will still have no term in x^1, so for $f\in V+W$ we have $a_1=0$ as claimed. Conversely, if f has no term in x^1 then we can write

$$f(x)=a_0+a_2x^2+a_3x^3+a_4x^4=a_3x^3+(a_0+a_2x^2+a_4x^4),$$

with $a_3x^3\in V$ and $a_0+a_2x^2+a_4x^4\in W$, so $f\in V+W$.

In particular, the polynomial $f(x)=x$ does not lie in $V+W$, so $V+W\neq U$.

Definition 4.19. Let U and V be vector spaces, and let $\phi\colon U\to V$ be a linear map. Then we write

$$\begin{aligned}\ker(\phi)&=\{u\in U\mid \phi(u)=0\}\\ \text{image}(\phi)&=\{v\in V\mid v=\phi(u)\text{ for some }u\in U\}.\end{aligned}$$

Example 4.20. Define $\pi\colon\mathbb{R}^3\to\mathbb{R}^3$ by $\pi\left[\begin{smallmatrix}x\\y\\z\end{smallmatrix}\right]=\left[\begin{smallmatrix}0\\y\\z\end{smallmatrix}\right]$. Then

$$\begin{aligned}\ker(\pi)&=\{\left[\begin{smallmatrix}x\\0\\0\end{smallmatrix}\right]\mid x\in\mathbb{R}\}\\ \text{image}(\pi)&=\{\left[\begin{smallmatrix}0\\y\\z\end{smallmatrix}\right]\mid y,z\in\mathbb{R}\}\end{aligned}$$

Proposition 4.21. *Let U and V be vector spaces, and let $\phi\colon U\to V$ be a linear map. Then* $\ker(\phi)$ *is a subspace of U, and* $\text{image}(\phi)$ *is a subspace of V.*

Proof. Firstly, we have $\phi(0_U)=0_V$, which shows both that $0_U\in\ker(\phi)$ and that $0_V\in\text{image}(\phi)$. Next, suppose that $u,u'\in\ker(\phi)$, which means that $\phi(u)=\phi(u')=0$. As ϕ is linear this implies that $\phi(u+u')=\phi(u)+\phi(u')=0+0=0$, so $u+u'\in\ker(\phi)$. This shows that $\ker(\phi)$ is closed under addition. Now suppose we have $t\in\mathbb{R}$. Using the linearity of ϕ again, we have $\phi(tu)=t\phi(u)=t.0=0$, so $tu\in\ker(\phi)$. This means that $\ker(\phi)$ is also closed under scalar multiplication, so it is a subspace of U. Now suppose we have $v,v'\in\text{image}(\phi)$. This means that we can find $x,x'\in U$ with $\phi(x)=v$ and $\phi(x')=v'$. We thus have $x+x',tx\in U$ and as ϕ is linear we have $\phi(x+x')=\phi(x)+\phi(x')=v+v'$ and $\phi(tx)=t\phi(x)=tv$. This shows that $v+v'$ and tv lie in $\text{image}(\phi)$, so $\text{image}(\phi)$ is closed under addition and scalar multiplication, so it is a subspace. ∎

Example 4.22. Define $\phi\colon \mathbb{R}^3 \to \mathbb{R}^3$ by $\phi([x,y,z]^T) = [x-y, y-z, z-x]^T$. Then

$$\ker(\phi) = \{[x,y,z]^T \in \mathbb{R}^3 \mid x = y = z\} = \{[t,t,t]^T \mid t \in \mathbb{R}\}$$
$$\text{image}(\phi) = \{[x,y,z]^T \in \mathbb{R}^3 \mid x+y+z = 0\} = \{[x,y,-x-y]^T \mid x,y \in \mathbb{R}^2\}.$$

Indeed, for the kernel we have $[x,y,z]^T \in \ker(\phi)$ iff $\phi([x,y,z]^T) = [0,0,0]^T$ iff $x-y = y-z = z-x = 0$ iff $x = y = z$, which means that $[x,y,z]^T = [t,t,t]^T$ for some t.

For the image, note that if $x+y+z = 0$ then

$$\phi([0,-x,-x-y]^T) = [0-(-x), (-x)-(-x-y), -x-y-0]^T = [x,y,z]^T,$$

so $[x,y,z]^T \in \text{image}(\phi)$. Conversely, if $[x,y,z]^T \in \text{image}(\phi)$ then $[x,y,z]^T = \phi([u,v,w]^T)$ for some $u,v,w \in \mathbb{R}$, which means that $x = u-v$ and $y = v-w$ and $z = w-u$, so

$$x+y+z = (u-v)+(v-w)+(w-u) = 0.$$

Thus $\ker(\phi)$ is a line through the origin (and thus a one-dimensional subspace) and $\text{image}(\phi)$ is a plane through the origin (and thus a two-dimensional subspace).

Example 4.23. Define $\phi\colon M_n\mathbb{R} \to M_n\mathbb{R}$ by $\phi(A) = A - A^T$ (which is linear). Then clearly $\phi(A) = 0$ iff $A = A^T$ iff A is a symmetric matrix. Thus

$$\ker(\phi) = \{n \times n \text{ symmetric matrices }\}.$$

We claim that also

$$\text{image}(\phi) = \{n \times n \text{ antisymmetric matrices }\}.$$

For brevity, we write W for the set of antisymmetric matrices, so we must show that $\text{image}(\phi) = W$. For any A we have $\phi(A)^T = (A - A^T)^T = A^T - A^{TT}$, but $A^{TT} = A$, so $\phi(A)^T = A^T - A = -\phi(A)$. This shows that $\phi(A)$ is always antisymmetric, so $\text{image}(\phi) \subseteq W$. Next, if B is antisymmetric then $B^T = -B$ so $\phi(B/2) = B/2 - B^T/2 = B/2 + B/2 = B$. Thus B is ϕ(something), so $B \in \text{image}(\phi)$. This shows that $W \subseteq \text{image}(\phi)$, so $W = \text{image}(\phi)$ as claimed.

Example 4.24. Define $\phi\colon \mathbb{R}[x]_{\leq 1} \to \mathbb{R}^3$ by $\phi(f) = [f(0), f(1), f(2)]^T$. More explicitly, we have

$$\phi(ax+b) = [b, a+b, 2a+b]^T = a[0,1,2]^T + b[1,1,1]^T.$$

If $ax+b \in \ker(\phi)$ then we must have $\phi(ax+b) = 0$, or in other words $b = a+b = 2a+b = 0$, which implies that $a = b = 0$ and so $ax+b = 0$. This means that $\ker(\phi) = \{0\}$.

Next, we claim that

$$\text{image}(\phi) = \{[u,v,w]^T \mid u - 2v + w = 0\}.$$

Indeed, if $[u,v,w]^T \in \text{image}(\phi)$ then we must have $[u,v,w] = \phi(ax+b) = [b, a+b, 2a+b]$ for some $a,b \in \mathbb{R}$. This means that $u - 2v + w = b - 2(a+b) + 2a + b = 0$, as required. Conversely, suppose that we have a vector $[u,v,w]^T \in \mathbb{R}^3$ with $u - 2v + w = 0$. We then have $w = 2v - u$ and so

$$\phi((v-u)x+u) = \begin{bmatrix} u \\ (v-u)+u \\ 2(v-u)+u \end{bmatrix} = \begin{bmatrix} u \\ v \\ 2v-u \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \end{bmatrix},$$

so $[u,v,w]^T$ is in the image of ϕ.

Remark 4.25. How did we arrive at our description of the image, and our proof that that description is correct? We need to consider a vector $[u,v,w]^T$ and ask whether it can be written as $\phi(f)$ for some polynomial $f(x) = ax+b$. In other words, we want to have $[u,v,w]^T = [b, a+b, 2a+b]^T$, which means that $u = b$ and $v = a+b$ and $w = 2a+b$. The first two equations tell us that the only possible solution is to take $a = v-u$ and $b = u$, so $f(x) = (v-u)x+u$. This potential solution is only a real solution if the third equation $w = 2a+b$ is also satisfied, which means that $w = 2(v-u)+u = 2v-u$, which means that $w - 2v + u = 0$.

Example 4.26. Define $\phi\colon \mathbb{R}[x]_{\leq 2} \to \mathbb{R}^2$ by $\phi(f) = [f(1), f'(1)]^T$. More explicitly, we have

$$\phi(ax^2+bx+c) = [a+b+c, 2a+b]^T.$$

It follows that ax^2+bx+c lies in $\ker(\phi)$ iff $a+b+c = 0 = 2a+b$, which gives $b = -2a$ and $c = -a-b = -a+2a = a$, so

$$ax^2+bx+c = ax^2 - 2ax + a = a(x^2-2x+1) = a(x-1)^2.$$

It follows that $\ker(\phi) = \{a(x-1)^2 \mid a \in \mathbb{R}\}$. In particular, $\ker(\phi)$ is nonzero, so ϕ is not injective. Explicitly, we have $x^2+1 \neq 2x$ but $\phi(x^2+1) = [2,2]^T = \phi(2x)$.

On the other hand, we claim that ϕ is surjective. Indeed, for any vector $\mathbf{a} = [u,v]^T \in \mathbb{R}^2$ we check that

$$\phi(vx+u-v) = [v+u-v, v]^T = [u,v]^T = \mathbf{a},$$

so $\mathbf{a}$ is ϕ(something) as required.

Remark 4.27. How did we arrive at the proof of surjectivity? We need to find a polynomial $f(x) = ax^2 + bx + c$ such that $\phi(f) = [u, v]^T$, or equivalently $[a + b + c, 2a + b] = [u, v]$, which means that $a + b + c = u$ and $2a + b = v$. These equations can be solved to give $b = v - 2a$ and $c = u - v + a$, with a arbitrary. We can choose to take $a = 0$, giving $b = v$ and $c = u - v$, so $f(x) = vx + u - v$.

Recall that a map $\phi\colon U \to V$ is *surjective* if every element $v \in V$ has the form $\phi(u)$ for some $u \in U$. Moreover, ϕ is said to be *injective* if whenever $\phi(u) = \phi(u')$ we have $u = u'$.

Proposition 4.28. *Let U and V be vector spaces, and let $\phi\colon U \to V$ be a linear map. Then ϕ is injective iff* $\ker(\phi) = \{0\}$*, and ϕ is surjective iff* $\text{image}(\phi) = V$.

Proof.
- Suppose that ϕ is injective, so whenever $\phi(u) = \phi(u')$ we have $u = u'$. Suppose that $u \in \ker(\phi)$. Then $\phi(u) = 0 = \phi(0)$. As ϕ is injective and $\phi(u) = \phi(0)$, we must have $u = 0$. Thus $\ker(\phi) = \{0\}$, as claimed.
- Conversely, suppose that $\ker(\phi) = \{0\}$. Suppose that $\phi(u) = \phi(u')$. Then $\phi(u - u') = \phi(u) - \phi(u') = 0$, so $u - u' \in \ker(\phi) = \{0\}$, so $u - u' = 0$, so $u = u'$. This means that ϕ is injective.
- Recall that $\text{image}(\phi)$ is the set of those $v \in V$ such that $v = \phi(u)$ for some $u \in U$. Thus $\text{image}(\phi) = V$ iff *every* element $v \in V$ has the form $\phi(u)$ for some $u \in U$, which is precisely what it means for ϕ to be surjective.

□

Corollary 4.29. *$\phi\colon U \to V$ is an isomorphism iff* $\ker(\phi) = 0$ *and* $\text{image}(\phi) = V$. □

Example 4.30. Consider the map $\phi\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$\phi([x, y, z]^T) = [x - y, y - z, z - x]^T$$

as in Example 4.22. Then $\ker(\phi) = \{[t, t, t]^T \mid t \in \mathbb{R}\}$, which is not zero, so ϕ is not injective. Explicitly, we have $[1, 2, 3]^T \neq [4, 5, 6]^T$ but

$$\phi([1, 2, 3]^T) = [1 - 2, 2 - 3, 3 - 1]^T = [-1, -1, 2]^T = \phi([4, 5, 6]^T),$$

so $[1, 2, 3]^T$ and $[4, 5, 6]^T$ are distinct points with the same image under ϕ, so ϕ is not injective. Moreover, we have seen that

$$\text{image}(\phi) = \{[u, v, w]^T \mid u + v + w = 0\},$$

which is not all of $\mathbb{R}^3$. In particular, the vector $[1, 1, 1]^T$ does not lie in $\text{image}(\phi)$ (because $1 + 1 + 1 \neq 0$), so it cannot be written as $\phi([x, y, z]^T)$ for any $[x, y, z]$. This means that ϕ is not surjective.

Example 4.31. Consider the map $\phi\colon \mathbb{R}[x]_{\leq 1} \to \mathbb{R}^3$ given by $\phi(f) = [f(0), f(1), f(2)]^T$ as in Example 4.24. We saw there that $\ker(\phi) = \{0\}$, so ϕ is injective. However, we have $\text{image}(\phi) = \{[u, v, w]^T \in \mathbb{R}^3 \mid u - 2v + w = 0\}$, which is not the whole of $\mathbb{R}^3$. In particular, the vector $\mathbf{a} = [1, 0, 0]^T$ does not lie in $\text{image}(\phi)$ (because $1 - 2.0 + 0 \neq 0$), so ϕ is not surjective.

Example 4.32. Consider the map $\phi\colon \mathbb{R}[x]_{\leq 2} \to \mathbb{R}^2$ given by $\phi(f) = [f(1), f'(1)]^T$, as in Example 4.26. We saw there that the polynomial $f(x) = (x - 1)^2$ is a nonzero element of $\ker(\phi)$, so ϕ is not injective. We also saw that $\text{image}(\phi) = \mathbb{R}^2$, so ϕ is surjective.

Definition 4.33. Let V and W be vector spaces. We define $V \oplus W$ to be the set of pairs (v, w) with $v \in V$ and $w \in W$. Addition and scalar multiplication are defined in the obvious way:

$$(v, w) + (v', w') = (v + v', w + w')$$
$$t.(v, w) = (tv, tw).$$

This makes $V \oplus W$ into a vector space, called the *direct sum* of V and W. We may sometimes use the notation $V \times W$ instead of $V \oplus W$.

Example 4.34. An element of $\mathbb{R}^p \oplus \mathbb{R}^q$ is a pair $(\mathbf{x}, \mathbf{y})$, where $\mathbf{x}$ is a list of p real numbers, and $\mathbf{y}$ is a list of q real numbers. Such a pair is essentially the same thing as a list of $p + q$ real numbers, so $\mathbb{R}^p \oplus \mathbb{R}^q = \mathbb{R}^{p+q}$.

Remark 4.35. Strictly speaking, $\mathbb{R}^p \oplus \mathbb{R}^q$ is only isomorphic to $\mathbb{R}^{p+q}$, not equal to it. This is a pedantic distinction if you are doing things by hand, but it becomes more significant if you are using a computer. Maple would represent an element of $\mathbb{R}^2 \oplus \mathbb{R}^3$ as something like `a=[<10,20>,<7,8,9>]`, and an element of $\mathbb{R}^5$ as something like `b=<10,20,7,8,9>`. To convert from the second form to the first, you can use syntax like this:

```
a := [b[1..2],b[3..5]];
```

Conversion the other way is a little more tricky. It is easiest to define an auxiliary function called `strip()` as follows:

```
strip := (u) -> op(convert(u,list));
```

This converts a vector (like `<7,8,9>`) to an unbracketed sequence (like `7,8,9`). You can then do

```
b := < strip(a[1]), strip(a[2]) >;
```

Now suppose that V and W are subspaces of a third space U. We then have a space $V \oplus W$ as above, and also a subspace $V + W \leq U$ as in Definition 4.13. We need to understand the relationship between these.

Proposition 4.36. *The rule $\sigma(v, w) = v + w$ defines a linear map $\sigma\colon V \oplus W \to U$, whose image is $V + W$, and whose kernel is the space $X = \{(x, -x) \in V \oplus W \mid x \in V \cap W\}$. Thus, if $V \cap W = 0$ then $\ker(\sigma) = 0$ and σ gives an isomorphism $V \oplus W \to V + W$.*

Proof. We leave it as an exercise to check that σ is a linear map. The image is the set of things of the form $v + w$ for some $v \in V$ and $w \in W$, which is precisely the definition of $V + W$. The kernel is the set of pairs $(x, y) \in V \oplus W$ for which $x + y = 0$. This means that $x \in V$ and $y \in W$ and $y = -x$. Note then that $x = -y$ and $y \in W$ so $x \in W$. We also have $x \in V$, so $x \in V \cap W$. This shows that $\ker(\sigma) = \{(x, -x) \mid x \in V \cap W\}$, as claimed. If $V \cap W = 0$ then we get $\ker(\sigma) = 0$, so σ is injective (by Proposition 4.28). If we regard it as a map to $V + W$ (rather than to U) then it is also surjective, so it is an isomorphism $V \oplus W \to V + W$, as claimed. □

Remark 4.37. If $V \cap W = 0$ and $V + W = U$ then σ gives an isomorphism $V \oplus W \to U$. In this situation it is common to say that $U = V \oplus W$. This is not strictly true (because U is only isomorphic to $V \oplus W$, not equal to it), but it is a harmless abuse of language. Sometimes people call $V \oplus W$ the *external direct sum* of V and W, and they say that U is the *internal direct sum* of V and W if $U = V + W$ and $V \cap W = 0$.

Example 4.38. Consider the space F of all functions from $\mathbb{R}$ to $\mathbb{R}$, and the subspaces EF and OF of even functions and odd functions. We claim that $F = EF \oplus OF$. To prove this, we must check that $EF \cap OF = 0$ and $EF + OF = F$. Suppose that $f \in EF \cap OF$. Then for any x we have $f(x) = f(-x)$ (because $f \in EF$), but $f(-x) = -f(x)$ (because $f \in OF$), so $f(x) = -f(x)$, so $f(x) = 0$. Thus $EF \cap OF = 0$, as required. Next, consider an arbitrary function $g \in F$. Put

$$\begin{aligned} g_+(x) &= (g(x) + g(-x))/2 \\ g_-(x) &= (g(x) - g(-x))/2. \end{aligned}$$

Then

$$\begin{aligned} g_+(-x) &= (g(-x) + g(x))/2 = g_+(x) \\ g_-(-x) &= (g(-x) - g(x))/2 = -g_-(x), \end{aligned}$$

so $g_+ \in EF$ and $g_- \in OF$. It is also clear from the formulae that $g = g_+ + g_-$, so $g \in EF + OF$. This shows that $EF + OF = F$, so $F = EF \oplus OF$ as claimed.

Example 4.39. Put

$$\begin{aligned} U &= M_2\mathbb{R} \\ V &= \{A \in M_2\mathbb{R} \mid \operatorname{trace}(A) = 0\} = \{\left[\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right] \mid a, b, c \in \mathbb{R}\} \\ W &= \{tI \mid t \in \mathbb{R}\} = \{\left[\begin{smallmatrix} t & 0 \\ 0 & t \end{smallmatrix}\right] \mid t \in \mathbb{R}\}. \end{aligned}$$

We claim that $U = V \oplus W$. To check this, first suppose that $A \in V \cap W$. As $A \in W$ we have $A = tI$ for some $t \in \mathbb{R}$, but $\operatorname{trace}(A) = 0$ (because $A \in V$) whereas $\operatorname{trace}(tI) = 2t$, so we must have $t = 0$, which means that $A = 0$. This shows that $V \cap W = 0$. Next, consider an arbitrary matrix $B = \left[\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right] \in U$. We can write this as $B = C + D$, where

$$\begin{aligned} C &= \left[\begin{smallmatrix} (p-s)/2 & q \\ r & (s-p)/2 \end{smallmatrix}\right] \in V \\ D &= \left[\begin{smallmatrix} (p+s)/2 & 0 \\ 0 & (p+s)/2 \end{smallmatrix}\right] = \frac{p+s}{2} I \in W. \end{aligned}$$

This shows that $U = V + W$.

Remark 4.40. How did we find these formulae? We have a matrix $B = \left[\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right] \in U$, and we want to write it as $B = C + D$ with $C \in U$ and $D \in W$. We must then have $C = \left[\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right]$ for some a, b, c and $D = \left[\begin{smallmatrix} t & 0 \\ 0 & t \end{smallmatrix}\right]$ for some t, and we want to have

$$\left[\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right] = \left[\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right] + \left[\begin{smallmatrix} t & 0 \\ 0 & t \end{smallmatrix}\right] = \left[\begin{smallmatrix} t+a & b \\ c & t-a \end{smallmatrix}\right],$$

so $b = q$ and $c = r$ and $t + a = p$ and $t - a = s$, which gives $a = (p - s)/2$ and $t = (p + s)/2$ as before.

5. Independence and spanning sets

Two randomly-chosen vectors in $\mathbb{R}^2$ will generally not be parallel; it is an important special case if they happen to point in the same direction.

Similarly, given three vectors u, v and w in $\mathbb{R}^3$, there will usually not be any plane that contains all three vectors. This means that we can get from the origin to any point by travelling a certain (possibly negative) distance in the direction of u, then a certain distance in the direction of v, then a certain distance in the direction of w. The case where

u, v and w all lie in a common plane will have special geometric significance in any purely mathematical problem, and will often have special physical significance in applied problems.

Our task in this section is to generalise these ideas, and study the corresponding special cases in an arbitrary vector space V. The abstract picture will be illuminating even in the case of $\mathbb{R}^2$ and $\mathbb{R}^3$.

Definition 5.1. Let V be a vector space, and let $\mathcal{V} = v_1, \dots, v_n$ be a list of elements of V. A *linear relation* between the v_i's is a vector $[\lambda_1, \dots, \lambda_n]^T \in \mathbb{R}^n$ such that $\lambda_1 v_1 + \dots + \lambda_n v_n = 0$. The vector $[0, \dots, 0]^T$ is obviously a linear relation, called the *trivial relation.* If there is a nontrivial linear relation, we say that the list $\mathcal{V}$ is *linearly dependent.* Otherwise, if the only relation is the trivial one, we say that the list $\mathcal{V}$ is *linearly independent.*

Example 5.2. Consider the following vectors in $\mathbb{R}^3$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

Then one finds that $\mathbf{v}_1 - 2\mathbf{v}_2 + \mathbf{v}_3 = 0$, so $[1, -2, 1]^T$ is a nontrivial linear relation, so the list $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is linearly dependent.

Example 5.3. Consider the following vectors:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

A linear relation between these is a vector $[\lambda_1, \lambda_2, \lambda_3]^T$ such that $\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = 0$, or equivalently

$$\begin{bmatrix} \lambda_1 \\ \lambda_1 + \lambda_2 \\ \lambda_1 + \lambda_2 + \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

From this we see that $\lambda_1 = 0$, then from the equation $\lambda_1 + \lambda_2 = 0$ we see that $\lambda_2 = 0$, then from the equation $\lambda_1 + \lambda_2 + \lambda_3 = 0$ we see that $\lambda_3 = 0$. Thus, the only linear relation is the trivial one where $[\lambda_1, \lambda_2, \lambda_3] = [0, 0, 0]$, so our vectors v_1, v_2, v_3 are linearly independent.

Example 5.4. Consider the polynomials $p_n(x) = (x + n)^2$, so

$$\begin{aligned} p_0(x) &= x^2 \\ p_1(x) &= x^2 + 2x + 1 \\ p_2(x) &= x^2 + 4x + 4 \\ p_3(x) &= x^2 + 6x + 9. \end{aligned}$$

I claim that the list p_0, p_1, p_2 is linearly independent. Indeed, a linear relation between them is a vector $[\lambda_0, \lambda_1, \lambda_2]^T$ such that $\lambda_0 p_0 + \lambda_1 p_1 + \lambda_2 p_2 = 0$, or equivalently

$$(\lambda_0 + \lambda_1 + \lambda_2)x^2 + (2\lambda_1 + 4\lambda_2)x + (\lambda_1 + 4\lambda_2) = 0$$

for all x, or equivalently

$$\lambda_0 + \lambda_1 + \lambda_2 = 0, \qquad 2\lambda_1 + 4\lambda_2 = 0, \qquad \lambda_1 + 4\lambda_2 = 0.$$

Subtracting the last two equations gives $\lambda_1 = 0$, putting this in the last equation gives $\lambda_2 = 0$, and now the first equation gives $\lambda_0 = 0$. Thus, the only linear relation is $[\lambda_0, \lambda_1, \lambda_2]^T = [0, 0, 0]^T$, so the list p_0, p_1, p_2 is independent.

I next claim, however, that the list p_0, p_1, p_2, p_3 is linearly dependent. Indeed, you can check that $p_3 - 3p_2 + 3p_1 - p_0 = 0$, so $[1, -3, 3, -1]^T$ is a nontrivial linear relation. (The entries in this list are the coefficients in the expansion of $(T - 1)^3 = T^3 - 3T^2 + 3T - 1$; this is not a coincidence, but the explanation would take us too far afield.)

Example 5.5. Consider the functions

$$\begin{aligned} f_1(x) &= e^x \\ f_2(x) &= e^{-x} \\ f_3(x) &= \sinh(x) \\ f_4(x) &= \cosh(x). \end{aligned}$$

These are linearly dependent, because $\sinh(x)$ is by definition just $(e^x - e^{-x})/2$, so

$$f_1 - f_2 - 2f_3 = e^x - e^{-x} - (e^x - e^{-x}) = 0,$$

so $[1, -1, 2, 0]^T$ is a nontrivial linear relation. Similarly, we have $\cosh(x) = (e^x + e^{-x})/2$, so $[1, 1, 0, -2]^T$ is another linear relation.

Example 5.6. Consider the matrices

$$E_1 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right] \qquad E_2 = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] \qquad E_3 = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right] \qquad E_4 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right].$$

A linear relation between these is a vector $[\lambda_1, \lambda_2, \lambda_3, \lambda_4]^T$ such that $\lambda_1 E_1 + \lambda_2 E_2 + \lambda_3 E_3 + \lambda_4 E_4$ is the zero matrix. But

$$\lambda_1 E_1 + \lambda_2 E_2 + \lambda_3 E_3 + \lambda_4 E_4 = \left[\begin{smallmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{smallmatrix}\right],$$

and this is only the zero matrix if $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$. Thus, the only linear relation is the trivial one, showing that $E_1, \ldots, E_4$ are linearly independent.

Remark 5.7. Let V be a vector space, and let $\mathcal{V} = v_1, \ldots, v_n$ be a list of elements of V. We have a linear map $\mu_{\mathcal{V}} \colon \mathbb{R}^n \to V$, given by

$$\mu_{\mathcal{V}}([\lambda_1, \ldots, \lambda_n]^T) = \lambda_1 v_1 + \ldots + \lambda_n v_n.$$

By definition, a linear relation between the v_i's is just a vector $\lambda = [\lambda_1, \ldots, \lambda_n]^T \in \mathbb{R}^n$ such that $\mu_{\mathcal{V}}(\lambda) = 0$, or in other words, an element of the kernel of $\mu_{\mathcal{V}}$. Thus, $\mathcal{V}$ is linearly independent iff $\ker(\mu_{\mathcal{V}}) = \{0\}$ iff $\mu_{\mathcal{V}}$ is injective (by Proposition 4.28).

Definition 5.8. Let $C^\infty(\mathbb{R})$ be the vector space of smooth functions $f \colon \mathbb{R} \to \mathbb{R}$. Given $f_1, \ldots, f_n \in C^\infty(\mathbb{R})$, their *Wronskian matrix* is the matrix $WM(f_1, \ldots, f_n)$ whose entries are the derivatives $f_i^{(j)}$ for $i = 1, \ldots, n$ and $j = 0, \ldots, n-1$. For example, in the case $n = 4$, we have

$$WM(f_1, f_2, f_3, f_4) = \begin{bmatrix} f_1 & f_2 & f_3 & f_4 \\ f_1' & f_2' & f_3' & f_4' \\ f_1'' & f_2'' & f_3'' & f_4'' \\ f_1''' & f_2''' & f_3''' & f_4''' \end{bmatrix}.$$

The *Wronskian* of $f_1, \ldots, f_n$ is the determinant of the Wronskian matrix; it is written $W(f_1, \ldots, f_n)$. Note that the entries in the Wronskian matrix are all functions, so the determinant is again a function.

Example 5.9. Consider the functions exp and sin and cos, so $\exp' = \exp$ and $\sin' = \cos$ and $\cos' = -\sin$ and $\sin^2 + \cos^2 = 1$. We have

$$W(\exp, \sin, \cos) = \det \begin{bmatrix} \exp & \sin & \cos \\ \exp & \sin' & \cos' \\ \exp & \sin'' & \cos'' \end{bmatrix} = \det \begin{bmatrix} \exp & \sin & \cos \\ \exp & \cos & -\sin \\ \exp & -\sin & -\cos \end{bmatrix}$$
$$= \exp.(-\cos^2 - \sin^2) - \exp.(-\sin.\cos + \sin.\cos) + \exp.(-\sin^2 - \cos^2) = -2\exp.$$

Proposition 5.10. *If $f_1, \ldots, f_n$ are linearly dependent, then $W(f_1, \ldots, f_n) = 0$. (More precisely, the function $w = W(f_1, \ldots, f_n)$ is the zero function, ie $w(x) = 0$ for all x.)*

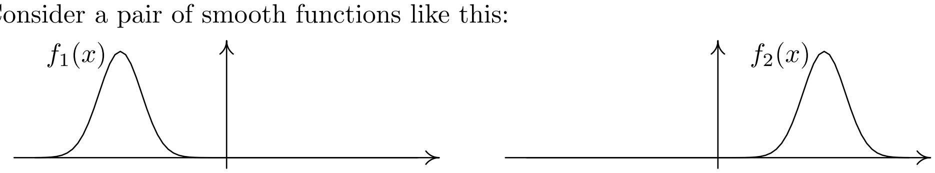
Proof. We will prove the case $n = 3$. The general case is essentially the same, but it just needs more complicated notation. If f_1, f_2, f_3 are linearly dependent, then there are numbers $\lambda_1, \lambda_2, \lambda_3$ (not all zero) such that $\lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3$ is the zero function, which means that

$$\lambda_1 f_1(x) + \lambda_2 f_2(x) + \lambda_3 f_3(x) = 0$$

for all x. We can differentiate this identity to see that $\lambda_1 f_1'(x) + \lambda_2 f_2'(x) + \lambda_3 f_3'(x) = 0$ for all x, and then differentiate again to see that $\lambda_1 f_1''(x) + \lambda_2 f_2''(x) + \lambda_3 f_3''(x) = 0$ for all x. Thus, if we let $\mathbf{u}_i$ be the vector $\left[\begin{smallmatrix} f_i(x) \\ f_i'(x) \\ f_i''(x) \end{smallmatrix}\right]$ (which is the i'th column of the Wronskian matrix), we see that $\lambda_1 \mathbf{u}_1 + \lambda_2 \mathbf{u}_2 + \lambda_3 \mathbf{u}_3 = 0$. This means that the columns of the Wronskian matrix are linearly dependent, which means that the determinant is zero, as claimed. □

Corollary 5.11. *If $W(f_1, \ldots, f_n) \neq 0$, then $f_1, \ldots, f_n$ are linearly independent.* □

Remark 5.12. Consider a pair of smooth functions like this:



Suppose that $f_1(x)$ is zero (not just small) for $x \geq 0$, and that $f_2(x)$ is zero for $x \leq 0$. (It is not easy to write down formulae for such functions, but it can be done; we will not discuss this further here.) For $x \leq 0$, the matrix $WM(f_1, f_2)(x)$ has the form $\begin{bmatrix} f_1(x) & 0 \\ f_1'(x) & 0 \end{bmatrix}$, so the determinant is zero. For $x \geq 0$, the matrix $WM(f_1, f_2)(x)$ has the form $\begin{bmatrix} 0 & f_2(x) \\ 0 & f_2'(x) \end{bmatrix}$, so the determinant is again zero. Thus $W(f_1, f_2)(x) = 0$ for all x, but f_1 and f_2 are not linearly dependent. This shows that the test in Proposition 5.10 is not reversible: if the functions are dependent then the Wronskian vanishes, but if the Wronskian vanishes then the functions need not be dependent. In practice it is rare to find such counterexamples, however.

Definition 5.13. Given a list $\mathcal{V} = v_1, \ldots, v_n$ of elements of a vector space V, we write $\text{span}(\mathcal{V})$ for the set of all vectors $w \in V$ that can be written in the form $w = \lambda_1 v_1 + \ldots + \lambda_n v_n$ for some $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. Equivalently, $\text{span}(\mathcal{V})$ is the image of the map $\mu_{\mathcal{V}} \colon \mathbb{R}^n \to V$ (which shows that $\text{span}(\mathcal{V})$ is a subspace of V). We say that $\mathcal{V}$ *spans* V if $\text{span}(\mathcal{V}) = V$, or equivalently, if $\mu_{\mathcal{V}}$ is surjective.

Remark 5.14. Often V will be a subspace of some larger space U. If you are asked whether certain vectors $v_1, \ldots, v_n$ span V, the *first* thing that you have to check is that they are actually elements of V.

There is an obvious spanning list for $\mathbb{R}^n$.

Definition 5.15. Let $\mathbf{e}_i$ be the vector in $\mathbb{R}^n$ whose i'th entry is 1, with all other entries being zero. For example, in $\mathbb{R}^3$ we have

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Example 5.16. The list $\mathbf{e}_1, \ldots, \mathbf{e}_n$ spans $\mathbb{R}^n$. Indeed, any vector $\mathbf{x} \in \mathbb{R}^n$ can be written as $x_1\mathbf{e}_1 + \ldots + x_n\mathbf{e}_n$, which is a linear combination of $\mathbf{e}_1, \ldots, \mathbf{e}_n$, as required. For example, in $\mathbb{R}^3$ we have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3.$$

Example 5.17. The list $1, x, \ldots, x^n$ spans $\mathbb{R}[x]_{\leq n}$. Indeed, any element of $\mathbb{R}[x]_{\leq n}$ is a polynomial of the form $f(x) = a_0 + a_1 x + \cdots + a_n x^n$, and so is visibly a linear combination of $1, x, \ldots, x^n$.

Example 5.18. Consider the vectors

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{u}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad \mathbf{u}_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

We claim that these span $\mathbb{R}^4$. Indeed, consider an arbitrary vector $\mathbf{v} = \begin{bmatrix} a & b & c & d \end{bmatrix}^T \in \mathbb{R}^4$. We have

$$(a-c+d)\mathbf{u}_1 + (c-d)\mathbf{u}_2 + (c-a)\mathbf{u}_3 + (b-c)\mathbf{u}_3 = \begin{bmatrix} a-c+d \\ a-c+d \\ a-c+d \\ a-c+d \end{bmatrix} + \begin{bmatrix} c-d \\ c-d \\ c-d \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ c-a \\ c-a \\ c-a \end{bmatrix} + \begin{bmatrix} 0 \\ b-c \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \mathbf{v},$$

which shows that $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \ldots, \mathbf{u}_4$, as required.

This is a perfectly valid argument, but it does rely on a formula that we pulled out of a hat. Here is an explanation of how the formula was constructed. We want to find p, q, r, s such that $\mathbf{v} = p\mathbf{u}_1 + q\mathbf{u}_2 + r\mathbf{u}_3 + s\mathbf{u}_4$, or equivalently

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = p \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + q \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + r \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} p+q \\ p+q+r+s \\ p+q+r \\ p+r \end{bmatrix}, \text{ or}$$

$$p+q = a \ (1) \qquad p+q+r+s = b \ (2) \qquad p+q+r = c \ (3) \qquad p+r = d \ (4)$$

Subtracting (3) and (4) gives $q = c - d$; subtracting (1) and(3) gives $r = c - a$; subtracting (2) and(3) gives $s = b - c$; putting $q = c - d$ in (1) gives $p = a - c + d$. With these values we have

$$(a-c+d)\mathbf{u}_1 + (c-d)\mathbf{u}_2 + (c-a)\mathbf{u}_3 + (b-c)\mathbf{u}_3 = \begin{bmatrix} a-c+d \\ a-c+d \\ a-c+d \\ a-c+d \end{bmatrix} + \begin{bmatrix} c-d \\ c-d \\ c-d \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ c-a \\ c-a \\ c-a \end{bmatrix} + \begin{bmatrix} 0 \\ b-c \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \mathbf{v}$$

as required.

Example 5.19. Consider the polynomials $p_i(x) = (x+i)^2$. We claim that the list $p_{-2}, p_{-1}, p_0, p_1, p_2$ spans $\mathbb{R}[x]_{\leq 2}$. Indeed, we have

$$\begin{aligned} p_0(x) &= x^2 \\ p_1(x) - p_{-1}(x) &= (x+1)^2 - (x-1)^2 = 4x \\ p_2(x) + p_{-2}(x) - 2p_0(x) &= (x+2)^2 + (x-2)^2 - 2x^2 = 8. \end{aligned}$$

Thus for an arbitrary quadratic polynomial $f(x) = ax^2 + bx + c$, we have

$$\begin{aligned} f(x) &= ap_0(x) + \tfrac{1}{4}b(p_1(x) - p_{-1}(x)) + \tfrac{1}{8}c(p_2(x) + p_{-2}(x) - 2p_0(x)) \\ &= \tfrac{c}{8}p_{-2}(x) - \tfrac{b}{4}p_{-1}(x) + (a - \tfrac{c}{4})p_0(x) + \tfrac{b}{4}p_1(x) + \tfrac{c}{8}p_2(x). \end{aligned}$$

Example 5.20. Put $V = \{f \in C^\infty(\mathbb{R}) \mid f'' + f = 0\}$. We claim that the functions sin and cos span V. In other words, we claim that if f is a solution to the equation $f''(x) = -f(x)$ for all x, then there are constants a and b such that $f(x) = a\sin(x) + b\cos(x)$ for all x. You have probably heard in a differential equations course that this is true, but you may not have seen a proof, so we will give one.

Firstly, we have $\sin' = \cos$ and $\cos' = -\sin$, so $\sin'' = -\sin$ and $\cos'' = -\cos$, so sin and cos are indeed elements of V. Consider an arbitrary element $f \in V$. Put $a = f'(0)$ and $b = f(0)$, and put $g(x) = f(x) - a\sin(x) - b\cos(x)$. We claim that $g = 0$. First, we have

$$\begin{aligned} g(0) &= f(0) - a\sin(0) - b\cos(0) = b - a.0 - b.1 = 0 \\ g'(0) &= f'(0) - a\sin'(0) - b\cos'(0) = a - a\cos(0) + b\sin(0) = a - a.1 - b.0 = 0. \end{aligned}$$

Now put $h(x) = g(x)^2 + g'(x)^2$; the above shows that $h(0) = 0$. Next, we have $g \in V$, so $g'' = -g$, so

$$h'(x) = 2g(x)g'(x) + 2g'(x)g''(x) = 2g'(x)(g(x) + g''(x)) = 0.$$

This means that h is constant, but $h(0) = 0$, so $h(x) = 0$ for all x. However, $h(x) = g(x)^2 + g'(x)^2$, which is the sum of two nonnegative quantities; the only way we can have $h(x) = 0$ is if $g(x) = 0 = g'(x)$. This means that $g = 0$, so $f(x) - a\sin(x) - b\cos(x) = 0$, so $f(x) = a\sin(x) + b\cos(x)$, as required.

This argument has an interesting physical interpretation. You should think of $g(x)$ as representing some kind of vibration. The term $g(x)^2$ gives the elastic energy and $g'(x)^2$ gives the kinetic energy, so the equation $h'(x) = 0$ is just conservation of total energy.

Definition 5.21. A vector space V is *finite-dimensional* if there is a (finite) list $\mathcal{V} = v_1, \dots, v_n$ of elements of V that spans V.

Example 5.22. Using our earlier examples of spanning sets, we see that the spaces $\mathbb{R}^n$, $M_{n,m}\mathbb{R}$ and $\mathbb{R}[x]_{\leq n}$ are finite-dimensional.

Example 5.23. The space $\mathbb{R}[x]$ is not finite-dimensional. To see this, consider a list $\mathcal{P} = p_1, \dots, p_n$ of polynomials. Let d be the maximum of the degrees of $p_1, \dots, p_n$. Then p_i lies in $\mathbb{R}[x]_{\leq d}$ for all i, so the span of $\mathcal{P}$ is contained in $\mathbb{R}[x]_{\leq d}$. In particular, the polynomial x^{d+1} does not lie in $\operatorname{span}(\mathcal{P})$, so $\mathcal{P}$ does not span all of $\mathbb{R}[x]$.

Definition 5.24. A *basis* for a vector space V is a list $\mathcal{V}$ of elements of V that is linearly independent and also spans V. Equivalently, a list $\mathcal{V} = v_1, \dots, v_n$ is a basis iff the map $\mu_{\mathcal{V}} \colon \mathbb{R}^n \to V$ is a bijection.

Example 5.25. We will find a basis for the space V of antisymmetric 3×3 matrices. Such a matrix has the form

$$X = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}.$$

In other words, if we put

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix},$$

then any antisymmetric matrix X can be written in the form $X = aA + bB + cC$. This means that the matrices A, B and C span V, and they are clearly independent, so they form a basis.

Example 5.26. Put $V = \{A \in M_3R \mid A^T = A \text{ and } \operatorname{trace}(A) = 0\}$. Any matrix $X \in V$ has the form

$$X = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & -a-d \end{bmatrix}$$

for some $a, b, c, d, e \in \mathbb{R}$. In other words, if we put

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

then any matrix $X \in V$ can be written in the form

$$X = aA + bB + cC + dD + eE.$$

This means that the matrices $A, \dots, E$ span V, and they are also linearly independent, so they form a basis for V.

Example 5.27. There are several interesting bases for the space $Q = \mathbb{R}[x]_{\leq 2}$ of polynomials of degree at most two. A typical element $f \in Q$ has $f(x) = ax^2 + bx + c$ for some $a, b, c \in \mathbb{R}$.

- The list p_0, p_1, p_2, where $p_i(x) = x^i$. This is the most obvious basis. For f as above we have
$$f = c\,p_0 + b\,p_1 + a\,p_2 = f(0)\,p_0 + f'(0)\,p_1 + \tfrac{1}{2}f''(0)p_2.$$
- The list q_0, q_1, q_2, where $q_i(x) = (x+1)^i$, is another basis. For f as above, one checks that
$$ax^2 + bx + c = a(x+1)^2 + (b - 2a)(x+1) + (a - b + c)$$
so $f = (a - b + c)q_0 + (b - 2a)q_1 + a\,q_2 = f(-1)q_0 + f'(-1)q_1 + \frac{1}{2}f''(-1)q_2$.

- The list r_0, r_1, r_2, where $r_i(x) = (x+i)^2$, is another basis. Indeed, we have

$$\begin{aligned} p_0(x) &= 1 = \tfrac{1}{2}((x+2)^2 - 2(x+1)^2 + x^2) \\ &= \tfrac{1}{2}(r_2(x) - 2r_1(x) + r_0(x)) \\ p_1(x) &= x = -\tfrac{1}{4}((x+2)^2 - 4(x+1)^2 + 3x^2) \\ &= -\tfrac{1}{4}(r_2(x) - 4r_1(x) + 3r_0(x)) \\ p_2(x) &= x^2 = r_0(x). \end{aligned}$$

  This implies that $p_0, p_1, p_2 \in \operatorname{span}(r_0, r_1, r_2)$ and thus that $\operatorname{span}(r_0, r_1, r_2) = Q$.
- The list

$$\begin{aligned} s_0(x) &= (x^2 - 3x + 2)/2 \\ s_1(x) &= -x^2 + 2x \\ s_2(x) &= (x^2 - x)/2. \end{aligned}$$

  These functions have the property that

$$\begin{matrix} s_0(0) = 1 & s_0(1) = 0 & s_0(2) = 0 \\ s_1(0) = 0 & s_1(1) = 1 & s_1(2) = 0 \\ s_2(0) = 0 & s_2(1) = 0 & s_2(2) = 1 \end{matrix}$$

  Given $f \in Q$ we claim that $f = f(0).s_0 + f(1).s_1 + f(2).s_2$. Indeed, if we put $g(x) = f(x) - f(0)s_0(x) - f(1)s_1(x) - f(2).s_2(x)$, we find that $g \in Q$ and $g(0) = g(1) = g(2) = 0$. A quadratic polynomial with three different roots must be zero, so $g = 0$, so $f = f(0).s_0 + f(1).s_1 + f(2).s_2$.
- The list

$$\begin{aligned} t_0(x) &= 1 \\ t_1(x) &= \sqrt{3}(2x - 1) \\ t_2(x) &= \sqrt{5}(6x^2 - 6x + 1). \end{aligned}$$

  These functions have the property that

$$\int_0^1 t_i(x)t_j(x)\,dx = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

  Using this, we find that $f = \lambda_0 t_0 + \lambda_1 t_1 + \lambda_2 t_2$, where $\lambda_i = \int_0^1 f(x)t_i(x)\,dx$.

Example 5.28. Put $V = \{f \in \mathbb{R}[x]_{\leq 4} \mid f(1) = f(-1) = 0 \text{ and } f'(1) = f'(-1)\}$. Consider a polynomial $f \in \mathbb{R}[x]_{\leq 4}$, so $f(x) = a + bx + cx^2 + dx^3 + ex^4$
for some constants $a, \ldots, e$. We then have

$$\begin{aligned} f(1) &= a + b + c + d + e \\ f(-1) &= a - b + c - d + e \\ f'(1) - f'(-1) &= (b + 2c + 3d + 4e) - (b - 2c + 3d - 4e) = 4c + 8e \end{aligned}$$

It follows that $f \in V$ iff $a + b + c + d + e = a - b + c - d + e = 4c + 8e = 0$.
This simplifies to $c = -2e$ and $a = e$ and $b = -d$, so

$$f(x) = e - dx - 2ex^2 + dx^3 + ex^4 = d(x^3 - x) + e(x^4 - 2x^2 + 1).$$

Thus, if we put $p(x) = x^3 - x$ and $q(x) = x^4 - 2x^2 + 1 = (x^2 - 1)^2$, then p, q is a basis for V.

Example 5.29. A *magic square* is a 3×3 matrix in which the sum of every row is the same, and the sum of every column is the same. More explicitly, a matrix

$$X = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

is a magic square iff we have

$$\begin{aligned} a + b + c &= d + e + f = g + h + i \\ a + d + g &= b + e + h = c + f + i. \end{aligned}$$

Let V be the set of magic squares, which is easily seen to be a subspace of $M_3\mathbb{R}$; we will find a basis for V.

First, we write

$$\begin{aligned} R(X) &= a+b+c = d+e+f = g+h+i \\ C(X) &= a+d+g = b+e+h = c+f+i \\ T(X) &= a+b+c+d+e+f+g+h+i. \end{aligned}$$

on the one hand, we have

$$T(X) = a+b+c+d+e+f+g+h+i = (a+b+c)+(d+e+f)+(g+h+i) = 3R(X).$$

We also have

$$T(X) = a+d+g+b+e+h+c+f+i = (a+d+g)+(b+e+h)+(c+f+i) = 3C(X).$$

It follows that $R(X) = C(X) = T(X)/3$.

It is now convenient to consider the subspace $W = \{X \in V \mid T(X) = 0\}$, consisting of squares as above for which

$$\begin{aligned} a+b+c = d+e+f = g+h+i = 0 \\ a+d+g = b+e+h = c+f+i = 0. \end{aligned}$$

For such a square, we certainly have

$$\begin{aligned} c &= -a-b \\ f &= -d-e \\ g &= -a-d \\ h &= -b-e. \end{aligned}$$

Substituting this back into the equation $g+h+i=0$ (or into the equation $c+f+i=0$) gives $i = a+b+d+e$. It follows that any element of W can be written in the form

$$X = \begin{bmatrix} a & b & -a-b \\ d & e & -d-e \\ -a-d & -b-e & a+b+d+e \end{bmatrix}.$$

Equivalently, if we put

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix} \qquad E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix},$$

then any element of W can be written in the form

$$X = aA + bB + dD + eE$$

for some list a, b, d, e of real numbers. This means that A, B, D, E spans W, and these matrices are clearly linearly independent, so they form a basis for W.

Next, observe that the matrix

$$Q = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

lies in V but not in W (because $T(Q) = 9$). We claim that Q, A, B, D, E is a basis for V. Indeed, given $X \in V$ we can put $t = T(X)/9$ and $Y = X - tQ$. We then have $Y \in V$ and $T(Y) = T(X) - tT(Q) = 0$, so $Y \in W$. As A, B, D, E is a basis for W, we see that $Y = aA + bB + dD + eE$ for some $a, b, d, e \in \mathbb{R}$. It follows that $X = tQ + Y = tQ + aA + bB + dD + eE$. This means that Q, A, B, D, E spans V.

Suppose we have a linear relation

$$qQ + aA + bB + dD + eE = 0$$

for some $q, a, b, d, e \in \mathbb{R}$. Applying T to this equation gives $9q = 0$ (because $T(A) = T(B) = T(D) = T(E) = 0$ and $T(Q) = 9$), and so $q = 0$. This leaves $aA + bB + dD + eE = 0$, and we have noted that A, B, D and E are linearly independent, so $a = b = d = e = 0$ as well. This means that Q, A, B, D and E are linearly independent as well as spanning V, so they form a basis for V. Thus $\dim(V) = 5$.

6. Linear maps out of $\mathbb{R}^n$

We next discuss linear maps $\mathbb{R}^n \to V$ (for any vector space V). We will do the case $n = 2$ first; the general case is essentially the same, but with more complicated notation.

Definition 6.1. Let V be a vector space, and let v and w be elements of V. We then define $\mu_{v,w}\colon \mathbb{R}^2 \to V$ by

$$\mu_{v,w}\left(\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]\right) = xv + yw.$$

This makes sense because:

- x is a number and $v \in V$ and V is a vector space, so $xv \in V$.
- y is a number and $w \in V$ and V is a vector space, so $yw \in V$.
- xv and yw lie in the vector space V, so $xv + yw \in V$.

It is clear that $\mu_{v,w}$ is a linear map.

Proposition 6.2. *Any linear map $\phi\colon \mathbb{R}^2 \to V$ has the form $\phi = \mu_{v,w}$ for some $v, w \in V$.*

Proof. The vector $\mathbf{e}_1 = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ is an element of $\mathbb{R}^2$, and ϕ is a map from $\mathbb{R}^2$ to V, so we have an element $v = \phi(\mathbf{e}_1) \in V$. Similarly, the vector $\mathbf{e}_2 = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ is an element of $\mathbb{R}^2$, and ϕ is a map from $\mathbb{R}^2$ to V, so we have an element $w = \phi(\mathbf{e}_2) \in V$. We claim that $\phi = \mu_{v,w}$. Indeed, as ϕ is linear, we have

$$\begin{aligned} \phi(x\mathbf{e}_1 + y\mathbf{e}_2) &= x\phi(\mathbf{e}_1) + y\phi(\mathbf{e}_2) \\ &= xv + yw \\ &= \mu_{v,w}(x, y). \end{aligned}$$

On the other hand, it is clear that

$$x\mathbf{e}_1 + y\mathbf{e}_2 = x\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] + y\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right],$$

so the previous equation reads

$$\phi\left(\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]\right) = \mu_{v,w}\left(\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]\right).$$

This holds for all x and y, so $\phi = \mu_{v,w}$ as claimed. □

The story for general n is as follows. Recall that for any list $\mathcal{V} = v_1, \dots, v_n$ of elements of V, we can define a linear map $\mu_{\mathcal{V}}\colon \mathbb{R}^n \to V$ by

$$\mu_{\mathcal{V}}([x_1, \dots, x_n]^T) = \sum_i x_i v_i = x_1 v_1 + \ldots + x_n v_n.$$

Proposition 6.3. *Any linear map $\phi\colon \mathbb{R}^n \to V$ has the form $\phi = \mu_{\mathcal{V}}$ for some list $\mathcal{V} = v_1, \dots, v_n$ of elements of V (which are uniquely determined by the formula $v_i = \phi(\mathbf{e}_i)$, where $\mathbf{e}_i$ is as in Definition 5.15). Thus, a linear map $\mathbb{R}^n \to V$ is essentially the same thing as a list of n elements of V.*

Proof. Put $v_i = \phi(\mathbf{e}_i) \in V$. For any $\mathbf{x} \in \mathbb{R}^n$ we have

$$\mathbf{x} = x_1\mathbf{e}_1 + \ldots + x_n\mathbf{e}_n = \sum_i x_i \mathbf{e}_i,$$

so

$$\phi(\mathbf{x}) = \sum_i x_i \phi(\mathbf{e}_i) = \sum_i x_i v_i = \mu_{v_1,\dots,v_n}(\mathbf{x}),$$

so $\phi = \mu_{v_1,\dots,v_n}$. (The first equality holds because ϕ is linear, the second by the definition of v_i, and the third by the definition of $\mu_{\mathcal{V}}$. □

Example 6.4. Consider the map $\phi\colon \mathbb{R}^3 \to M_3\mathbb{R}$ given by

$$\phi\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a & a+b & a \\ a+b & a+b+c & a+b \\ a & a+b & a \end{bmatrix}$$

Put $\mathcal{A} = A_1, A_2, A_3$, where

$$A_1 = \phi(\mathbf{e}_1) = \begin{bmatrix} 1&1&1 \\ 1&1&1 \\ 1&1&1 \end{bmatrix} \quad A_2 = \phi(\mathbf{e}_2) = \begin{bmatrix} 0&1&0 \\ 1&1&1 \\ 0&1&0 \end{bmatrix} \quad A_3 = \phi(\mathbf{e}_3) = \begin{bmatrix} 0&0&0 \\ 0&1&0 \\ 0&0&0 \end{bmatrix}$$

Then

$$\mu_{\mathcal{A}}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\begin{bmatrix} 1&1&1 \\ 1&1&1 \\ 1&1&1 \end{bmatrix} + b\begin{bmatrix} 0&1&0 \\ 1&1&1 \\ 0&1&0 \end{bmatrix} + c\begin{bmatrix} 0&0&0 \\ 0&1&0 \\ 0&0&0 \end{bmatrix} = \begin{bmatrix} a & a+b & a \\ a+b & a+b+c & a+b \\ a & a+b & a \end{bmatrix} = \phi\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

so $\phi = \mu_{\mathcal{A}}$.

Example 6.5. Consider the map $\phi\colon \mathbb{R}^3 \to \mathbb{R}[x]$ given by

$$\phi\begin{bmatrix} a \\ b \\ c \end{bmatrix} = (a+b+c)x^2 + (a+b)(x+1)^2 + a(x+2)^2.$$

Put $\mathcal{P} = p_1, p_2, p_3$, where

$$\begin{aligned} p_1(x) &= \phi(\mathbf{e}_1) = x^2 + (x+1)^2 + (x+2)^2 = 3x^2 + 6x + 5 \\ p_2(x) &= \phi(\mathbf{e}_2) = x^2 + (x+1)^2 = 2x^2 + 2x + 1 \\ p_3(x) &= \phi(\mathbf{e}_3) = x^2. \end{aligned}$$

Then

$$\mu_{\mathcal{P}}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a(3x^2+6x+5) + b(2x^2+2x+1) + cx^2 = \phi\begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

Corollary 6.6. *Every linear map $\alpha\colon \mathbb{R}^n \to \mathbb{R}^m$ has the form ϕ_A (as in Example 3.11) for some $m\times n$ matrix A (which is uniquely determined). Thus, a linear map $\alpha\colon \mathbb{R}^n \to \mathbb{R}^m$ is essentially the same thing as an $m\times n$ matrix.*

Proof. A linear map $\alpha\colon \mathbb{R}^n \to \mathbb{R}^m$ is essentially the same thing as a list $\mathbf{v}_1, \dots, \mathbf{v}_n$ of elements of $\mathbb{R}^m$. If we write each $\mathbf{v}_i$ as a column vector, then the list can be visualised in an obvious way as an $m\times n$ matrix. For example, the list

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \end{bmatrix}$$

corresponds to the matrix

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{bmatrix}.$$

Thus, a linear map $\alpha\colon \mathbb{R}^n \to \mathbb{R}^m$ is essentially the same thing as an $m\times n$ matrix. There are some things to check to see that this is compatible with Example 3.11, but we shall not go through the details. □

Example 6.7. Consider the linear map $\rho\colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$\rho\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y \\ z \\ x \end{bmatrix}$$

(so $\rho(\mathbf{v})$ is obtained by rotating $\mathbf{v}$ through $2\pi/3$ around the line $x=y=z$). Then

$$\rho(\mathbf{e}_1) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \rho(\mathbf{e}_2) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \rho(\mathbf{e}_3) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

This means that $\rho = \phi_R$, where

$$R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Example 6.8. Consider a vector $\mathbf{a} = [a,b,c]^T \in \mathbb{R}^3$, and define $\beta\colon \mathbb{R}^3 \to \mathbb{R}^3$ by $\beta(\mathbf{v}) = \mathbf{a}\times\mathbf{v}$. This is linear, so it must have the form $\beta = \phi_B$ for some 3×3 matrix B. To find B, we note that

$$\beta\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} bz-cy \\ cx-az \\ ay-bx \end{bmatrix},$$

so

$$\beta(\mathbf{e}_1) = \begin{bmatrix} 0 \\ c \\ -b \end{bmatrix} \qquad \beta(\mathbf{e}_2) = \begin{bmatrix} -c \\ 0 \\ a \end{bmatrix} \qquad \beta(\mathbf{e}_3) = \begin{bmatrix} b \\ -a \\ 0 \end{bmatrix}.$$

These three vectors are the columns of B, so

$$B = \begin{bmatrix} 0 & -c & b \\ c & 0 & a \\ -b & a & 0 \end{bmatrix}.$$

(Note incidentally that the matrices arising in this way are precisely the 3×3 antisymmetric matrices.)

Example 6.9. Consider a unit vector $\mathbf{a} = [a,b,c]^T \in \mathbb{R}^3$ (so $a^2+b^2+c^2=1$) and let P be the plane perpendicular to $\mathbf{a}$. For any $\mathbf{v}\in\mathbb{R}^3$, we let $\pi(\mathbf{v})$ be the projection of $\mathbf{v}$ onto P. The formula for this is

$$\pi(\mathbf{v}) = \mathbf{v} - \langle \mathbf{v}, \mathbf{a}\rangle \mathbf{a}$$

(where $\langle \mathbf{v}, \mathbf{a}\rangle$ denotes the inner product, also written as $\mathbf{v}.\mathbf{a}$.) You can just take this as given if you are not familiar with it. From the formula one can check that π is a linear map, so it must have the form $\pi(\mathbf{v}) = A\mathbf{v}$ for some 3×3 matrix A. To find A, we observe that

$$\pi\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} - (ax+by+cz)\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} x-a^2x-aby-acz \\ y-abx-b^2y-bcz \\ z-acx-bcy-c^2z \end{bmatrix}.$$

It follows that

$$\pi(\mathbf{e}_1) = \begin{bmatrix} 1-a^2 \\ -ab \\ -ac \end{bmatrix} \qquad \pi(\mathbf{e}_2) = \begin{bmatrix} -ab \\ 1-b^2 \\ -bc \end{bmatrix} \qquad \pi(\mathbf{e}_3) = \begin{bmatrix} -ac \\ -bc \\ 1-c^2 \end{bmatrix}.$$

These three vectors are the columns of A, so

$$A = \begin{bmatrix} 1-a^2 & -ab & -ac \\ -ab & 1-b^2 & -bc \\ -ac & -bc & 1-c^2 \end{bmatrix}.$$

It is an exercise to check that $A^2 = A^T = A$ and $\det(A) = 0$.

7. Matrices for linear maps

Let V and W be finite-dimensional vector spaces, with bases $\mathcal{V} = v_1, \dots, v_n$ and $\mathcal{W} = w_1, \dots, w_m$ say. Let $\alpha\colon V \to W$ be a linear map. Then $\alpha(v_j)$ is an element of W, so it can be expressed (uniquely) in terms of the basis $\mathcal{W}$, say

$$\alpha(v_j) = a_{1j}w_1 + \cdots + a_{mj}w_m.$$

These numbers a_{ij} form an $m \times n$ matrix A, which we call *the matrix of α with respect to $\mathcal{V}$ and $\mathcal{W}$*.

Remark 7.1. Often we consider the case where $W = V$ and so we have a map $\alpha\colon V \to V$, and $\mathcal{V}$ and $\mathcal{W}$ are bases for the same space. It is often natural to take $\mathcal{W} = \mathcal{V}$, but everything still makes sense even if $\mathcal{W} \neq \mathcal{V}$.

Example 7.2. Let $\mathbf{a}$ be a unit vector in $\mathbb{R}^3$, and define $\beta, \pi\colon \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\beta(\mathbf{x}) = \mathbf{a} \times \mathbf{x}$$
$$\pi(\mathbf{x}) = \mathbf{x} - \langle \mathbf{a}, \mathbf{x} \rangle \mathbf{a}.$$

We have already calculated the matrices of these maps with respect to the standard basis of $\mathbb{R}^3$. However, it is sometimes useful to find a different basis that is specially suited to these particular maps, and find matrices with respect to that basis instead. To do this, choose any unit vector $\mathbf{b}$ orthogonal to $\mathbf{a}$, and then put $\mathbf{c} = \mathbf{a} \times \mathbf{b}$, so $\mathbf{c}$ is another unit vector that is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$. By standard properties of the cross product, we have $\beta(\mathbf{a}) = \mathbf{a} \times \mathbf{a} = 0$, and $\beta(\mathbf{b}) = \mathbf{a} \times \mathbf{b} = \mathbf{c}$ by definition of $\mathbf{c}$, and

$$\beta(\mathbf{c}) = \mathbf{a} \times (\mathbf{a} \times \mathbf{b}) = \langle \mathbf{a}, \mathbf{b} \rangle \mathbf{a} - \langle \mathbf{a}, \mathbf{a} \rangle \mathbf{b} = -\mathbf{b}.$$

In summary, we have

$$\begin{aligned} \beta(\mathbf{a}) &= 0 & &= 0\mathbf{a} + 0\mathbf{b} + 0\mathbf{c} \\ \beta(\mathbf{b}) &= \mathbf{c} & &= 0\mathbf{a} + 0\mathbf{b} + 1\mathbf{c} \\ \beta(\mathbf{c}) &= -\mathbf{b} & &= 0\mathbf{a} - 1\mathbf{b} + 0\mathbf{c}. \end{aligned}$$

The columns in the matrix we want are the lists of coefficients in the three equations above: the first equation gives the first column, the second equation gives the second column, and the third equation gives the third column. Thus, the the matrix of β with respect to the basis $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Similarly, we have $\pi(\mathbf{a}) = 0$ and $\pi(\mathbf{b}) = \mathbf{b}$ and $\pi(\mathbf{c}) = \mathbf{c}$, so the matrix of π with respect to the basis $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Example 7.3. Define $\phi\colon \mathbb{R}[x]_{<4} \to \mathbb{R}[x]_{<4}$ by $\phi(x^k) = (x+1)^k$. We then have

$$\begin{aligned} \phi(1) &= 1 \\ \phi(x) &= 1 + x \\ \phi(x^2) &= 1 + 2x + x^2 \\ \phi(x^3) &= 1 + 3x + 3x^2 + x^3, \end{aligned}$$

or in other words

$$\begin{aligned} \phi(x^0) &= 1.x^0 + 0.x^1 + 0.x^2 + 0.x^3 \\ \phi(x^1) &= 1.x^0 + 1.x^1 + 0.x^2 + 0.x^3 \\ \phi(x^2) &= 1.x^0 + 2.x^1 + 1.x^2 + 0.x^3 \\ \phi(x^3) &= 1.x^0 + 3.x^1 + 3.x^2 + 1.x^3. \end{aligned}$$

Thus, the matrix of ϕ with respect to the usual basis is

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Example 7.4. Define $\phi\colon \mathbb{R}[x]_{<5} \to \mathbb{R}^4$ by

$$\phi(f) = [f(1), f(2), f(3), f(4)]^T.$$

Then

$$\phi(1) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad \phi(x) = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \qquad \phi(x^2) = \begin{bmatrix} 1 \\ 4 \\ 9 \\ 16 \end{bmatrix} \qquad \phi(x^3) = \begin{bmatrix} 1 \\ 8 \\ 27 \\ 64 \end{bmatrix} \qquad \phi(x^4) = \begin{bmatrix} 1 \\ 16 \\ 81 \\ 256 \end{bmatrix}$$

so the matrix of ϕ with respect to the usual bases is

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 \\ 1 & 3 & 9 & 27 & 81 \\ 1 & 4 & 16 & 64 & 128 \end{bmatrix}.$$

Example 7.5. Define $\phi\colon \mathbb{R}^4 \to \mathbb{R}^4$ by

$$\phi \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_4 \\ x_3 \\ x_2 \\ x_1 \end{bmatrix}.$$

The associated matrix is

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Example 7.6. Let V be the space of solutions of the differential equation $f'' + f = 0$, and define $\phi\colon V \to V$ by $\phi(f)(x) = f(x + \pi/4)$. As

$$\sin(x + \pi/4) = \sin(x)\cos(\pi/4) + \cos(x)\sin(\pi/4) = (\sin(x) + \cos(x))/\sqrt{2},$$

we have $\phi(\sin) = (\sin + \cos)/\sqrt{2}$. As

$$\cos(x + \pi/4) = \cos(x)\cos(\pi/4) - \sin(x)\sin(\pi/4) = (\cos(x) - \sin(x))/\sqrt{2},$$

we have $\phi(\cos) = (-\sin + \cos)/\sqrt{2}$. It follows that the matrix of ϕ with respect to the basis $\{\sin, \cos\}$ is

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Example 7.7. Define $\phi, \psi\colon M_2\mathbb{R} \to M_2\mathbb{R}$ by $\phi(A) = A^T$ and $\psi(A) = A - \operatorname{trace}(A)I/2$. In terms of the usual basis

$$E_1 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right] \qquad E_2 = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] \qquad E_3 = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right] \qquad E_4 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$$

we have $\phi(E_1) = E_1$, $\phi(E_2) = E_3$, $\phi(E_3) = E_2$, and $\phi(E_4) = E_4$. The matrix of ϕ is thus

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We also have

$$\begin{aligned}
\psi(E_1) &= E_1 - I/2 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} = \tfrac{1}{2}E_1 - \tfrac{1}{2}E_4 \\
\psi(E_2) &= E_2 \\
\psi(E_3) &= E_3 \\
\psi(E_4) &= E_4 - I/2 = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = -\tfrac{1}{2}E_1 + \tfrac{1}{2}E_4.
\end{aligned}$$

The matrix is thus

$$\begin{bmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

The following result gives another important way to think about the matrix of a linear map.

Proposition 7.8. *For any $\mathbf{x} \in \mathbb{R}^n$, we have $\mu_{\mathcal{W}}(\phi_A(\mathbf{x})) = \alpha(\mu_{\mathcal{V}}(\mathbf{x}))$, so the two routes around the square below are the same:*

$$\begin{array}{ccc}
\mathbb{R}^n & \xrightarrow{\phi_A} & \mathbb{R}^m \\
{\scriptstyle \mu_{\mathcal{V}}}\big\downarrow & & \big\downarrow{\scriptstyle \mu_{\mathcal{W}}} \\
V & \xrightarrow[\alpha]{} & W
\end{array}$$

(This is often expressed by saying that the square commutes*.)*

Proof. We will do the case where $n = 2$ and $m = 3$; the general case is essentially the same, but with more complicated notation. In our case, v_1, v_2 is a basis for V, and w_1, w_2, w_3 is a basis for W. From the definitions of a_{ij} and A, we have

$$\alpha(v_1) = a_{11}w_1 + a_{21}w_2 + a_{31}w_3$$
$$\alpha(v_2) = a_{12}w_1 + a_{22}w_2 + a_{32}w_3$$
$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

Now consider a vector $\mathbf{x} = \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \in \mathbb{R}^2$. We have $\mu_{\mathcal{V}}(\mathbf{x}) = x_1v_1 + x_2v_2$ (by the definition of $\mu_{\mathcal{V}}$). It follows that

$$\begin{aligned} \alpha(\mu_{\mathcal{V}}(\mathbf{x})) &= \alpha(x_1v_1 + x_2v_2) = x_1\alpha(v_1) + x_2\alpha(v_2) \\ &= x_1(a_{11}w_1 + a_{21}w_2 + a_{31}w_3) + x_2(a_{12}w_1 + a_{22}w_2 + a_{32}w_3) \\ &= (a_{11}x_1 + a_{12}x_2)w_1 + (a_{21}x_1 + a_{22}x_2)w_2 + (a_{31}x_1 + a_{32}x_2)w_3 \end{aligned}$$

On the other hand, we have

$$\phi_A(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \\ a_{31}x_1 + a_{32}x_2 \end{bmatrix},$$

so

$$\begin{aligned} \mu_{\mathcal{W}}(\phi_A(\mathbf{x})) &= \mu_{\mathcal{W}}\left(\begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \\ a_{31}x_1 + a_{32}x_2 \end{bmatrix} \right) \\ &= (a_{11}x_1 + a_{12}x_2)w_1 + (a_{21}x_1 + a_{22}x_2)w_2 + (a_{31}x_1 + a_{32}x_2)w_3 \\ &= \alpha(\mu_{\mathcal{V}}(\mathbf{x})). \end{aligned}$$

□

Proposition 7.9. *Suppose we have linear maps $U \xrightarrow{\beta} V \xrightarrow{\alpha} W$ (which can therefore be composed to give a linear map $\alpha\beta\colon U \to W$). Suppose that we have bases $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ for U, V and W. Let A be the matrix of α with respect to $\mathcal{V}$ and $\mathcal{W}$, and let B be the matrix of β with respect to $\mathcal{U}$ and $\mathcal{V}$. Then the matrix of $\alpha\beta$ with respect to $\mathcal{U}$ and $\mathcal{W}$ is AB.*

Proof. By the definition of matrix multiplication, the matrix $C = AB$ has entries $c_{ik} = \sum_j a_{ij}b_{jk}$. By the definitions of A and B, we have

$$\alpha(v_j) = \sum_i a_{ij}w_i$$
$$\beta(u_k) = \sum_j b_{jk}v_j$$

so

$$\begin{aligned} \alpha\beta(u_k) &= \alpha\left(\sum_j b_{jk}v_j\right) = \sum_j b_{jk}\alpha(v_j) \\ &= \sum_j b_{jk}\sum_i a_{ij}w_i = \sum_i \left(\sum_j a_{ij}b_{jk}\right) w_i \\ &= \sum_i c_{ik}w_i. \end{aligned}$$

This means precisely that C is the matrix of $\alpha\beta$ with respect to $\mathcal{U}$ and $\mathcal{W}$. □

Definition 7.10. Let V be a finite-dimensional vector space, with two different bases $\mathcal{V} = v_1, \dots, v_n$ and $\mathcal{V}' = v'_1, \dots, v'_n$. We then have

$$v'_j = p_{1j}v_1 + \cdots + p_{nj}v_n$$

for some scalars p_{ij}. Let P be the $n \times n$ matrix with entries p_{ij}. This is called the *change-of-basis* matrix from $\mathcal{V}$ to $\mathcal{V}'$. One can check that it is invertible, and that P^{-1} is the change of basis matrix from $\mathcal{V}'$ to $\mathcal{V}$.

Example 7.11. Consider the following bases of $\mathbb{R}[x]_{\leq 3}$:

$$\begin{array}{llll} v_1 = x^3 & v_2 = x^2 & v_3 = x & v_4 = 1 \\ v'_1 = x^3+x^2+x+1 & v'_2 = x^3+x^2+x & v'_3 = x^3+x^2 & v'_4 = x^3 \end{array}$$

Then

$$\begin{aligned} v'_1 &= 1.v_1 + 1.v_2 + 1.v_3 + 1.v_4 \\ v'_2 &= 1.v_1 + 1.v_2 + 1.v_3 + 0.v_4 \\ v'_3 &= 1.v_1 + 1.v_2 + 0.v_3 + 0.v_4 \\ v'_4 &= 1.v_1 + 0.v_2 + 0.v_3 + 0.v_4 \end{aligned}$$

so the change of basis matrix is

$$P = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Example 7.12. Consider the following bases of $M_2\mathbb{R}$:

$$\begin{array}{llll} A_1 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right] & A_2 = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right] & A_3 = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right] & A_4 = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right] \\ A'_1 = \left[\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix}\right] & A'_2 = \left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right] & A'_3 = \left[\begin{smallmatrix} 1 & 1 \\ -1 & -1 \end{smallmatrix}\right] & A'_4 = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right] \end{array}$$

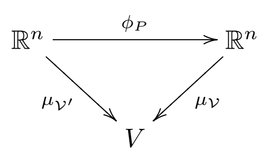
Then

$$\begin{aligned} A'_1 &= 2.A_1 + (-2).A_2 + 0.A_3 + 1.A_4 \\ A'_2 &= 0.A_1 + 0.A_2 + 2.A_3 + (-1).A_4 \\ A'_3 &= 0.A_1 + 2.A_2 + 0.A_3 + (-1).A_4 \\ A'_4 &= 0.A_1 + 0.A_2 + 0.A_3 + 1.A_4 \end{aligned}$$

so the change of basis matrix is

$$P = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Lemma 7.13. *In the situation above, for any $\mathbf{x} \in \mathbb{R}^n$ we have $\mu_{\mathcal{V}}(\phi_P(\mathbf{x})) = \mu_{\mathcal{V}}(P\mathbf{x}) = \mu_{\mathcal{V}'}(\mathbf{x})$, so the following diagram commutes:*



Proof. We have $P\mathbf{x} = \mathbf{y}$, where $y_i = \sum_j p_{ij} x_j$. Thus

$$\begin{aligned} \mu_{\mathcal{V}}(P\mathbf{x}) &= \sum_i y_i v_i = \sum_{i,j} p_{ij} x_j v_i \\ &= \sum_j x_j \left(\sum_i p_{ij} v_i \right) = \sum_j x_j v'_j \\ &= \mu_{\mathcal{V}'}(\mathbf{x}). \end{aligned}$$

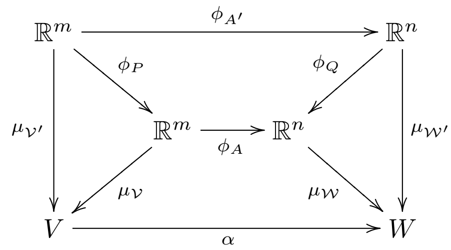
□

Proposition 7.14. *Let $\alpha \colon V \to W$ be a linear map between finite-dimensional vector spaces. Suppose we have two bases for V (say $\mathcal{V}$ and $\mathcal{V}'$, with change-of basis matrix P) and two bases for W (say $\mathcal{W}$ and $\mathcal{W}'$, with change-of-basis matrix Q). Let A be the matrix of α with respect to $\mathcal{V}$ and $\mathcal{W}$, and let A' be the matrix with respect to $\mathcal{V}'$ and $\mathcal{W}'$. Then $A' = Q^{-1}AP$.*

Proof. We actually prove that $QA' = AP$, which comes to the same thing. For any $\mathbf{x} \in \mathbb{R}^n$, we have

$$\begin{aligned} \mu_{\mathcal{W}}(QA'\mathbf{x}) &= \mu_{\mathcal{W}'}(A'\mathbf{x}) && \text{(Lemma 7.13)} \\ &= \alpha(\mu_{\mathcal{V}'}(\mathbf{x})) && \text{(Proposition 7.8)} \\ &= \alpha(\mu_{\mathcal{V}}(P\mathbf{x})) && \text{(Lemma 7.13)} \\ &= \mu_{\mathcal{W}}(AP\mathbf{x}) && \text{(Proposition 7.8).} \end{aligned}$$

This shows that $\mu_{\mathcal{W}}((QA'-AP)\mathbf{x})=0$. Moreover, $\mathcal{W}$ is linearly independent, so $\mu_{\mathcal{W}}$ is injective and has trivial kernel, so $(QA'-AP)\mathbf{x}=0$. This applies for *any* vector $\mathbf{x}$, so the matrix $QA'-AP$ must be zero, as claimed. The upshot is that all parts of the following diagram commute:



□

Remark 7.15. Suppose we have a finite-dimensional vector space V and a linear map α from V to itself. We can now define the trace, determinant and characteristic polynomial of α. We pick any basis $\mathcal{V}$, let A be the matrix of α with respect to $\mathcal{V}$ and $\mathcal{V}$, and put

$$\begin{aligned}\operatorname{trace}(\alpha)&=\operatorname{trace}(A)\\ \det(\alpha)&=\det(A)\\ \operatorname{char}(\alpha)(t)&=\operatorname{char}(A)(t)=\det(tI-A).\end{aligned}$$

This is not obviously well-defined: what if we used a different basis, say $\mathcal{V}'$, giving a different matrix, say A'? The proposition tells us that $P^{-1}AP=A'$, and it follows that $P^{-1}(tI-A)P=tI-A'$. Using the rules $\operatorname{trace}(MN)=\operatorname{trace}(NM)$ and $\det(MN)=\det(M)\det(N)$ we see that

$$\begin{aligned}\operatorname{trace}(A')&=\operatorname{trace}(P^{-1}(AP))=\operatorname{trace}((AP)P^{-1})=\operatorname{trace}(A(PP^{-1}))=\operatorname{trace}(A)\\ \det(A')&=\det(P)^{-1}\det(A)\det(P)=\det(A)\\ \operatorname{char}(A')(t)&=\det(P)^{-1}\det(tI-A)\det(P)=\operatorname{char}(A)(t).\end{aligned}$$

This shows that definitions are in fact basis-independent.

Example 7.16. Let $\mathbf{a}\in\mathbb{R}^3$ be a unit vector, and define $\beta\colon\mathbb{R}^3\to\mathbb{R}^3$ by $\beta(\mathbf{x})=\mathbf{a}\times\mathbf{x}$. The matrix B of β with respect to the standard basis is found as follows:

$$\beta(\mathbf{e}_1)=\begin{bmatrix}0\\a_3\\-a_2\end{bmatrix}\quad\beta(\mathbf{e}_2)=\begin{bmatrix}-a_3\\0\\a_1\end{bmatrix}\quad\beta(\mathbf{e}_3)=\begin{bmatrix}a_2\\-a_1\\0\end{bmatrix}\quad B=\begin{bmatrix}0&-a_3&a_2\\a_3&0&-a_1\\-a_2&a_1&0\end{bmatrix}$$

We have $\operatorname{trace}(B)=0$ and

$$\det(B)=0.\det\begin{bmatrix}0&-a_1\\a_1&0\end{bmatrix}-(-a_3).\det\begin{bmatrix}a_3&-a_1\\-a_2&0\end{bmatrix}+a_2.\det\begin{bmatrix}a_3&0\\-a_2&a_1\end{bmatrix}=0-(-a_3)(a_3.0-(-a_2)(-a_1))+a_2(a_3a_1-0.(-a_2))=0$$

We can insead choose a unit vector $\mathbf{b}$ orthogonal to $\mathbf{a}$ and then put $\mathbf{c}=\mathbf{a}\times\mathbf{b}$. With respect to the basis $\mathbf{a},\mathbf{b},\mathbf{c}$, the map β has matrix $B'=\begin{bmatrix}0&0&0\\0&0&-1\\0&1&0\end{bmatrix}$. It is easy to see that $\operatorname{trace}(B')=0=\det(B')$. Either way we have $\operatorname{trace}(\beta)=0=\det(\beta)$. We also find that $\operatorname{char}(\beta)(t)=\operatorname{char}(B')(t)=t^3+t$. This is much more complicated using B.

Example 7.17. Let $\mathbf{a}\in\mathbb{R}^3$ be a unit vector, and define $\pi\colon\mathbb{R}^3\to\mathbb{R}^3$ by $\pi(\mathbf{x})=\mathbf{x}-\langle\mathbf{x},\mathbf{a}\rangle\mathbf{a}$. The matrix P of π with respect to the standard basis is found as follows:

$$\pi(\mathbf{e}_1)=\begin{bmatrix}1-a_1^2\\-a_1a_2\\-a_1a_3\end{bmatrix}\quad\pi(\mathbf{e}_2)=\begin{bmatrix}-a_2a_1\\1-a_2^2\\-a_2a_3\end{bmatrix}\quad\pi(\mathbf{e}_3)=\begin{bmatrix}-a_3a_1\\-a_3a_2\\1-a_3^2\end{bmatrix}\quad P=\begin{bmatrix}1-a_1^2&-a_1a_2&-a_1a_3\\-a_1a_2&1-a_2^2&-a_2a_3\\-a_1a_3&-a_2a_3&1-a_3^2\end{bmatrix}$$

We have $\operatorname{trace}(P)=1-a_1^2+1-a_2^2+1-a_3^2=3-(a_1^2+a_2^2+a_3^2)=2$. We can instead choose a unit vector $\mathbf{b}$ orthogonal to $\mathbf{a}$ and then put $\mathbf{c}=\mathbf{a}\times\mathbf{b}$. With respect to the basis $\mathbf{a},\mathbf{b},\mathbf{c}$, the map π has matrix $P'=\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix}$. It is easy to see that $\operatorname{trace}(P')=2$. Either way we have $\operatorname{trace}(\pi)=2$. We also find that $\det(\pi)=\det(P')=0$ and $\operatorname{char}(\pi)(t)=\operatorname{char}(P')(t)=t(t-1)^2$. This is much more complicated using P.

Remark 7.18. Suppose again that we have a finite-dimensional vector space V and a linear map α from V to itself. One can show that the following are equivalent:

(a) α is injective
(b) α is surjective
(c) α is an isomorphism
(d) $\det(\alpha)\neq 0$.

(It is important here that α goes from V to itself, not to some other space.) We shall not give proofs, however.

8. Theorems about bases

For the next two results, we let V be a vector space, and let $\mathcal{V} = v_1, \dots, v_n$ be a list of elements in V. We put $V_i = \text{span}(v_1, \dots, v_i)$ (with the convention that $V_0 = 0$).

There may or may not be any nontrivial linear relations for $\mathcal{V}$. If there is a nontrivial relation λ, so that $\lambda_1 v_1 + \dots + \lambda_n v_n = 0$ and $\lambda_k \neq 0$ for some k, then we define the *height* of λ to be the largest i such that $\lambda_i \neq 0$. For example, if $n = 6$ and $5v_1 - 2v_2 - 2v_3 + 3v_4 = 0$ then $[5, -2, -2, 3, 0, 0]^T$ is a nontrivial linear relation of height 4.

Proposition 8.1. *The following are equivalent (so if any one of them is true, then so are the other two):*

(a) *The list $\mathcal{V}$ has a nontrivial linear relation of height i*
(b) $v_i \in V_{i-1}$
(c) $V_i = V_{i-1}$.

Example 8.2. Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \qquad v_4 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

Then $v_1 - 2v_2 + v_3 = 0$, so $[1, -2, 1, 0]^T$ is a linear relation of height 3. The equation can be rearranged as $v_3 = -v_1 + 2v_2$, showing that $v_3 \in \text{span}(v_1, v_2) = V_2$. One can check that

$$V_2 = V_3 = \{[x, y, z]^T \mid x + z = 2y\}.$$

Thus, in this example, with $i = 2$, we see that (a), (b) and (c) all hold.

Proof. (a)$\Rightarrow$(b): Let $\lambda = [\lambda_1, \dots, \lambda_n]^T$ be a nontrivial linear relation of height i, so $\lambda_1 v_1 + \ldots + \lambda_n v_n = 0$. The fact that the height is i means that $\lambda_i \neq 0$ but $\lambda_{i+1} = \lambda_{i+2} = \dots = 0$. We can thus rearrange the linear relation as

$$\begin{aligned} \lambda_i v_i &= -\lambda_1 v_1 - \dots - \lambda_{i-1} v_{i-1} - \lambda_{i+1} v_{i+1} - \dots - \lambda_n v_n \\ &= -\lambda_1 v_1 - \dots - \lambda_{i-1} v_{i-1} - 0.v_{i+1} - \dots - 0.v_n \\ &= -\lambda_1 v_1 - \dots - \lambda_{i-1} v_{i-1} \end{aligned}$$

so

$$v_i = -\lambda_1 \lambda_i^{-1} v_1 - \dots - \lambda_{i-1} \lambda_i^{-1} v_{i-1} \in V_{i-1}.$$

(b)$\Rightarrow$(a) Suppose that $v_i \in V_{i-1} = \text{span}(v_1, \dots, v_{i-1})$, so $v_i = \mu_1 v_1 + \dots + \mu_{i-1} v_{i-1}$ for some scalars $\mu_1, \dots, \mu_{i-1}$. We can rewrite this as a nontrivial linear relation

$$\mu_1 v_1 + \dots + \mu_{i-1} v_{i-1} + (-1).v_i + 0.v_{i+1} + \dots + 0.v_n = 0,$$

which clearly has height i.

(b)$\Rightarrow$(c): Suppose again that $v_i \in V_{i-1} = \text{span}(v_1, \dots, v_{i-1})$, so $v_i = \mu_1 v_1 + \dots + \mu_{i-1} v_{i-1}$ for some scalars $\mu_1, \dots, \mu_{i-1}$. We need to show that $V_i = V_{i-1}$, but it is clear that $V_{i-1} \leq V_i$, so it will be enough to show that $V_i \leq V_{i-1}$. Consider an element $w \in V_i$; we must show that $w \in V_{i-1}$. As $w \in V_i$ we have $w = \lambda_1 v_1 + \dots + \lambda_i v_i$ for some scalars $\lambda_1, \dots, \lambda_i$. This can be rewritten as

$$\begin{aligned} w =& \lambda_1 v_1 + \dots + \lambda_{i-1} v_{i-1} + \lambda_i(\mu_1 v_1 + \dots + \mu_{i-1} v_{i-1}) \\ =& (\lambda_1 + \lambda_i \mu_1) v_1 + (\lambda_2 + \lambda_i \mu_2) v_1 + \dots + (\lambda_{i-1} + \lambda_i \mu_{i-1}) v_{i-1}. \end{aligned}$$

This is a linear combination of $v_1, \dots, v_{i-1}$, showing that $w \in V_{i-1}$, as claimed.

(c)$\Rightarrow$(b): Suppose that $V_i = V_{i-1}$. It is clear that the element v_i lies in $\text{span}(v_1, \dots, v_i) = V_i$, but $V_i = V_{i-1}$, so $v_i \in V_{i-1}$. □

Corollary 8.3. *If for all i we have $v_i \notin V_{i-1}$, then there cannot be a linear relation of any height, so $\mathcal{V}$ must be linearly independent.* □

Corollary 8.4. *The following are equivalent:*

(a) *The list $\mathcal{V}$ has no nontrivial linear relation of height i*
(b) $v_i \notin V_{i-1}$ *(c) $V_i \neq V_{i-1}$.*

If these three things are true, we say that i is a *jump*.

Lemma 8.5. *Let V be a vector space, and let $\mathcal{V} = (v_1, \dots, v_n)$ be a finite list of elements of V that spans V. Then some sublist $\mathcal{V}' \subseteq \mathcal{V}$ is a basis for V.*

Proof. Let I' be the set of those integers $i \leq n$ for which $v_i \notin V_{i-1}$, and put $\mathcal{V}' = \{v_i \mid i \in I'\}$.

We first claim that $\mathcal{V}'$ is linearly independent. If not, then there is a nontrivial relation. If we write only the nontrivial terms, and keep them in the obvious order, then the relation takes the form $\lambda_{i_1} v_{i_1} + \dots + \lambda_{i_r} v_{i_r} = 0$ with $i_k \in I'$ for all k, and $\lambda_{i_k} \neq 0$ for all k, and $i_1 < \dots < i_r$. This can be regarded as a nontrivial linear relation for $\mathcal{V}$, of height i_r. Proposition 8.1 therefore tells us that $v_{i_r} \in V_{i_r - 1}$, which is impossible, as $i_r \in I'$. This contradiction shows that $\mathcal{V}'$ must be linearly independent, after all.

Now put $V' = \text{span}(\mathcal{V}')$. We will show by induction that $V_i \leq V'$ for all $i \leq n$. For the initial step, we note that $V_0 = 0$ so certainly $V_0 \leq V'$. Suppose that $V_{i-1} \leq V'$. There are two cases to consider:

(a) Suppose that $i \in I'$. Then (by the definition of $\mathcal{V}'$) we have $v_i \in \mathcal{V}'$ and so $v_i \in V'$. As $V_i = V_{i-1} + \mathbb{R}v_i$ and $V_{i-1} \leq V'$ and $\mathbb{R}v_i \leq V'$, we conclude that $V_i \leq V'$.

(b) Suppose that $i \notin I'$, so (by the definition of I') we have $v_i \in V_{i-1}$ and so $V_i = V_{i-1}$. By the induction hypothesis we have $V_{i-1} \leq V'$, so $V_i \leq V'$.

Either way we have $V_i \leq V'$, which proves the induction step. We therefore have $V_i \leq V'$ for all $i \leq n$. In particular, we have $V_n \leq V'$. However, V_n is just $\text{span}(\mathcal{V})$, and we assumed that $\mathcal{V}$ spans V, so $V_n = V$. This proves that $V \leq V'$, and it is clear that $V' \leq V$, so $V = V'$. This means that $\mathcal{V}'$ is a spanning list as well as being linearly independent, so it is a basis for V. □

Corollary 8.6. *Every finite-dimensional vector space has a basis.*

Proof. By Definition 5.21, we can find a finite list $\mathcal{V}$ that spans V. By Lemma 8.5, some sublist $\mathcal{V}' \subseteq \mathcal{V}$ is a basis. □

Lemma 8.7. *Let V be a vector space, and let $\mathcal{V} = (v_1, \dots, v_n)$ and $\mathcal{W} = (w_1, \dots, w_m)$ be finite lists of elements of $\mathcal{V}$ such that $\mathcal{V}$ spans V and $\mathcal{W}$ is linearly independent. Then $n \geq m$. (In other words, any spanning list is at least as long as any linearly independent list.)*

Proof. As before, we put $V_i = \text{span}(v_1, \dots, v_i)$, so $V_n = \text{span}(\mathcal{V}) = V$. We will show by induction that any linearly independent list in V_i has length at most i. In particular, this will show that any linearly independent list in $V = V_n$ has length at most n, as claimed.

For the initial step, note that $V_0 = 0$. This means that the only linearly independent list in V_0 is the empy list, which has length 0, as required.

Now suppose (for the induction step) that every linearly independent list in V_{i-1} has length at most $i-1$. Suppose we have a linearly independent list $(x_1, \dots, x_p)$ in V_i; we must show that $p \leq i$. The elements x_j lie in $V_i = \text{span}(v_1, \dots, v_i)$. We can thus find scalars a_{jk} such that

$$x_j = a_{j1} v_1 + a_{j2} v_2 + \dots + a_{j,i-1} v_{i-1} + a_{ji} v_i.$$

We need to consider two cases:

(a) Suppose that for each j the last coefficient a_{ji} is zero. This means that

$$x_j = a_{j1} v_1 + a_{j2} v_2 + \dots + a_{j,i-1} v_{i-1},$$

so $x_j \in \text{span}(v_1, \dots, v_{i-1}) = V_{i-1}$. This means that $x_1, \dots, x_p$ is a linearly independent list in V_{i-1}, so the induction hypothesis tells us that $p \leq i-1$, so certainly $p \leq i$.

(b) Otherwise, for some x_j we have $a_{ji} \neq 0$. It is harmless to reorder the x's, so for notational convenience we move this x_j to the end of the list, which means that $a_{pi} \neq 0$. Now put $\alpha_k = a_{ki} a_{pi}^{-1}$ and

$$y_k = x_k - \alpha_k x_p.$$

We will show that $y_1, \dots, y_{p-1}$ is a linearly independent list in V_{i-1}. Assuming this, the induction hypothesis gives $p - 1 \leq i - 1$, and so $p \leq i$ as required. First, we have

$$\begin{aligned} y_k =& x_k - a_{ki} a_{pi}^{-1} x_p \\ =& a_{k1} v_1 + \dots + a_{ki} v_i \\ & - a_{ki} a_{pi}^{-1} (a_{p1} v_1 + \dots + a_{pi} v_i) \\ =& (a_{k1} - a_{ki} a_{pi}^{-1} a_{p1}) v_1 + (a_{k2} - a_{ki} a_{pi}^{-1} a_{p2}) v_2 + \dots + (a_{ki} - a_{ki} a_{pi}^{-1} a_{pi}) v_i. \end{aligned}$$

In the last term, the coefficient $a_{ki} - a_{ki} a_{pi}^{-1} a_{pi}$ is zero, so y_k is actually a linear combination of $v_1, \dots, v_{i-1}$, so $y_k \in V_{i-1}$. Next, suppose we have a linear relation $\lambda_1 y_1 + \dots + \lambda_{p-1} y_{p-1} = 0$. Put

$$\lambda_p = -\lambda_1 \alpha_1 - \lambda_2 \alpha_2 - \dots - \lambda_{p-1} \alpha_{p-1}.$$

By putting $y_k = x_k - \alpha_k x_p$ in the relation $\lambda_1 y_1 + \cdots + \lambda_{p-1} y_{p-1} = 0$ and expanding it out, we get $\lambda_1 x_1 + \ldots + \lambda_{p-1} x_{p-1} + \lambda_p x_p = 0$. As $x_1, \ldots, x_p$ is linearly independent, this means that we must have $\lambda_1 = \cdots = \lambda_{p-1} = \lambda_p = 0$. It follows that our original relation among the y's was trivial. We conclude that the list $y_1, \ldots, y_{p-1}$ is a linearly independent list in V_{i-1}. As explained before, the induction hypothesis now tells us that $p - 1 \leq i - 1$, so $p \leq i$.

□

Corollary 8.8. *Let V be a finite-dimensional vector space. Then V has a finite basis, and any two bases have the same number of elements, say n. This number is called the* dimension *of V. Moreover, any spanning list for V has at least n elements, and any linearly independent list has at most n elements.*

Proof. We already saw in Corollary 8.6 that V has a basis, say $\mathcal{V} = (v_1, \ldots, v_n)$. Let $\mathcal{X}$ be a linearly independent listin V. As $\mathcal{V}$ is a spanning list and $\mathcal{X}$ is linearly independent, Lemma 8.7 tells us that $\mathcal{V}$ is at least as long as $\mathcal{X}$, so $\mathcal{X}$ has at most n elements. Now let $\mathcal{Y}$ be a spanning list for V. As $\mathcal{Y}$ spans and $\mathcal{V}$ is linearly independent, Lemma 8.7 tells us that $\mathcal{Y}$ is at least as long as $\mathcal{V}$, so $\mathcal{Y}$ has at least n elements. Now let $\mathcal{V}'$ be another basis for V. Then $\mathcal{V}'$ has at least n elements (because it spans) and at most n elements (because it is independent) so it must have exactly n elements. □

Corollary 8.9. *If V is a finite-dimensional vector space over K with dimension n, then we can choose a basis $\mathcal{V}$ of length n, and the map $\mu_{\mathcal{V}} \colon K^n \to V$ is an isomorphism, so K is isomorphic to K^n.*

Proposition 8.10. *Let V be a finite-dimensional vector space, and let W be a subspace of V. Then W is also finite-dimensional, and $\dim(W) \leq \dim(V)$.*

Proof. Put $n = \dim(V)$. We define a list $\mathcal{W} = (w_1, w_2, \ldots)$ as follows. If $W = 0$ then we take $\mathcal{W}$ to be the empty list. Otherwise, we let w_1 be any nonzero vector in W. If w_1 spans W we take $\mathcal{W} = (w_1)$. Otherwise, we can choose an element $w_2 \in W$ that is not in $\mathrm{span}(w_1)$. If $\mathrm{span}(w_1, w_2) = W$ then we stop and take $\mathcal{W} = (w_1, w_2)$. Otherwise, we can choose an element $w_3 \in W$ that is not in $\mathrm{span}(w_1, w_2)$. We continue in this way, so we always have $w_i \notin \mathrm{span}(w_1, \ldots, w_{i-1})$, so the w's are linearly independent (by Corollary 8.3). However, V has a spanning set of length n, so Lemma 8.7 tells us that we cannot have a linearly independent list of length greater than n, so our list of w's must stop before we get to w_{n+1}. This means that for some $p \leq n$ we have $W = \mathrm{span}(w_1, \ldots, w_p)$, so W is finite-dimensional, with $\dim(W) = p \leq n$. □

Proposition 8.11. *Let V be an n-dimensional vector space, and let $\mathcal{V} = (v_1, \ldots, v_p)$ be a linearly independent list of elements of V. Then $p \leq n$, and $\mathcal{V}$ can be extended to a list $\mathcal{V}' = (v_1, \ldots, v_n)$ such that $\mathcal{V}'$ is a basis of V.*

Proof. Corollary 8.8 tells us that $p \leq n$. If $\mathrm{span}(v_1, \ldots, v_p) = V$ then we take $\mathcal{V}' = (v_1, \ldots, v_p)$. Otherwise, we choose some $v_{p+1} \notin \mathrm{span}(v_1, \ldots, v_p)$. If $\mathrm{span}(v_1, \ldots, v_{p+1}) = V$ then we stop and take $\mathcal{V}' = (v_1, \ldots, v_{p+1})$. Otherwise, we choose some $v_{p+2} \notin \mathrm{span}(v_1, \ldots, v_{p+1})$ and continue in the same way. We always have $v_i \notin \mathrm{span}(v_1, \ldots, v_{i-1})$, so the v's are linearly independent (by Corollary 8.3). Any linearly independent list has length at most n (by Corollary 8.8) so our process must stop before we get to v_{n+1}. This means that $\mathcal{V}' = (v_1, \ldots, v_m)$ with $m \leq n$, and as the process has stopped, we must have $\mathrm{span}(\mathcal{V}') = V$. As $\mathcal{V}'$ is also linearly independent, we see that it is a basis, and so $m = n$ (by Corollary 8.8 again). □

Proposition 8.12. *Let V be an n-dimensional vector space.*

(a) *Any spanning list for V with exactly n elements is linearly independent, and so is a basis.*
(b) *Any linearly independent list in V with exactly n elements is a spanning list, and so is a basis.*

Proof.

(a) Let $\mathcal{V} = (v_1, \ldots, v_n)$ be a spanning list. Lemma 8.5 tells us that some sublist $\mathcal{V}' \subseteq \mathcal{V}$ is a basis for V. As $\dim(V) = n$, we see that $\mathcal{V}'$ has length n, but $\mathcal{V}$ also has length n, so $\mathcal{V}'$ must be all of $\mathcal{V}$. Thus, $\mathcal{V}$ itself must be a basis.
(b) Let $\mathcal{W} = (w_1, \ldots, w_n)$ be a linearly independent list. Proposition 8.11 tells us that $\mathcal{W}$ can be extended to a list $\mathcal{W}' \supseteq \mathcal{W}$ such that $\mathcal{W}'$ is a basis. In particular, $\mathcal{W}'$ must have length n, so it must just be the same as $\mathcal{W}$, so $\mathcal{W}$ itself is a basis.

□

Corollary 8.13. *Let V be an finite-dimensional vector space, and let W be a subspace with $\dim(W) = \dim(V)$; then $W = V$.*

Proof. Put $n = \dim(V) = \dim(V)$, and let $\mathcal{W} = (w_1, \ldots, w_n)$ be a basis for W. Then $\mathcal{W}$ is a linearly independent list in V with n elements, so part (b) of the Proposition tells us that $\mathcal{W}$ spans V. Thus $V = \mathrm{span}(\mathcal{W}) = W$. □

Proposition 8.14. *Let U be a finite-dimensional vector space, and let V and W be subspaces of U. Then one can find lists $(u_1, \ldots, u_p)$, $(v_1, \ldots, v_q)$ and $(w_1, \ldots, w_r)$ (for some $p, q, r \geq 0$) such that*

- $(u_1, \dots, u_p)$ *is a basis for* $V \cap W$
- $(u_1, \dots, u_p, v_1, \dots, v_q)$ *is a basis for* V
- $(u_1, \dots, u_p, w_1, \dots, w_r)$ *is a basis for* W
- $(u_1, \dots, u_p, v_1, \dots, v_q, w_1, \dots, w_r)$ *is a basis for* $V + W$.

In particular, we have

$$\begin{aligned} \dim(V \cap W) &= p \\ \dim(V) &= p + q \\ \dim(W) &= p + r \\ \dim(V + W) &= p + q + r, \end{aligned}$$

so $\dim(V) + \dim(W) = 2p + q + r = \dim(V \cap W) + \dim(V + W)$.

Proof. Choose a basis $\mathcal{U} = (u_1, \dots, u_p)$ for $V \cap W$. Then $\mathcal{U}$ is a linearly independent list in V, so it can be extended to a basis for V, say $(u_1, \dots, u_p, v_1, \dots, v_q)$. Similarly $\mathcal{U}$ is a linearly independent list in W, so it can be extended to a basis for W, say $(u_1, \dots, u_p, w_1, \dots, w_r)$. All that is left is to prove that the list

$$\mathcal{X} = (u_1, \dots, u_p, v_1, \dots, v_q, w_1, \dots, w_r)$$

is a basis for $V + W$. Consider an element

$$x = \alpha_1 u_1 + \dots + \alpha_p u_p + \beta_1 v_1 + \dots + \beta_q v_q + \gamma_1 w_1 + \dots + \gamma_r w_r \in \operatorname{span}(\mathcal{X}).$$

Put $y = \sum_i \alpha_i u_i + \sum_j \beta_j v_j$ and $z = \sum_k \gamma_k w_k$, so $x = y + z$. We have $u_i, v_j \in V$ and $w_k \in W$ so $y \in V$ and $z \in W$ so $x = y + z \in V + W$. Thus $\operatorname{span}(\mathcal{X}) \leq V + W$.

Now suppose we start with an element $x \in V + W$. We can then find $y \in V$ and $z \in W$ such that $x = y + z$. As $(u_1, \dots, u_p, v_1, \dots, v_q)$ is a basis for V, we have

$$y = \lambda_1 u_1 + \dots + \lambda_p u_p + \beta_1 v_1 + \dots + \beta_q v_q$$

for some scalars λ_i, β_j. Similarly, we have

$$z = \mu_1 u_1 + \dots + \mu_p u_p + \gamma_1 w_1 + \dots + \gamma_r w_r$$

for some scalars μ_i, γ_k. If we put $\alpha_i = \lambda_i + \mu_i$ we get

$$x = y + z = \alpha_1 u_1 + \dots + \alpha_p u_p + \beta_1 v_1 + \dots + \beta_q v_q + \gamma_1 w_1 + \dots + \gamma_r w_r \in \operatorname{span}(\mathcal{X}).$$

It follows that $\operatorname{span}(\mathcal{X}) = V + W$.

Finally, suppose we have a linear relation

$$\alpha_1 u_1 + \dots + \alpha_p u_p + \beta_1 v_1 + \dots + \beta_q v_q + \gamma_1 w_1 + \dots + \gamma_r w_r = 0.$$

We again put $y = \sum_i \alpha_i u_i + \sum_j \beta_j v_j$ and $z = \sum_k \gamma_k w_k$, so $y + z = 0$, so $z = -y$. Now $y \in V$, so z also lies in V, because $z = -y$. On the other hand, it is clear from our definition of z that it lies in W, so $z \in V \cap W$. We know that $\mathcal{U}$ is a basis for $V \cap W$, so $z = \lambda_1 u_1 + \dots + \lambda_p u_p$ for some $\lambda_1, \dots, \lambda_p$. This means that

$$\lambda_1 u_1 + \dots + \lambda_p u_p - \gamma_1 w_1 - \dots - \gamma_r w_r = 0.$$

We also know that $(u_1, \dots, u_p, w_1, \dots, w_r)$ is a basis for W, so the above equation implies that $\lambda_1 = \dots = \lambda_p = \gamma_1 = \dots = \gamma_r = 0$. Feeding this back into our original, relation, we get

$$\alpha_1 u_1 + \dots + \alpha_p u_p + \beta_1 v_1 + \dots + \beta_q v_q = 0.$$

However, we also know that $(u_1, \dots, u_p, v_1, \dots, v_q)$ is a basis for V, so the above equation implies that $\alpha_1 = \dots = \alpha_p = \beta_1 = \dots = \beta_q = 0$. As all α's, β's and γ's are zero, we see that our original linear relation was trivial. This shows that the list $\mathcal{X}$ is linearly independent, so it gives a basis for $V + W$ as claimed. □

Example 8.15. Put $U = M_3\mathbb{R}$ and

$$\begin{aligned} V &= \{A \in U \mid \text{ all rows sum to } 0 \ \} = \{A \in U \mid A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\} \\ W &= \{A \in U \mid \text{ all columns sum to } 0 \ \} = \{A \in U \mid [1,1,1]A = [0,0,0]\} \end{aligned}$$

Then $V \cap W$ is the set of all matrices of the form

$$A = \begin{bmatrix} a & b & -a-b \\ c & d & -c-d \\ -a-c & -b-d & a+b+c+d \end{bmatrix} = a \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix} + d \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

It follows that the list

$$u_1 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}, u_2 = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}, u_3 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}, u_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

is a basis for $V \cap W$. Now put

$$v_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}, w_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, w_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

so $v_i \in V$ and $w_i \in W$. A typical element of V has the form

$$\begin{aligned} A = \left[\begin{smallmatrix} a & b & -a-b \\ c & d & -c-d \\ e & f & -e-f \end{smallmatrix}\right] &= au_1 + bu_2 + cu_3 + du_4 + \left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e-a-c & f-b-d & a+b+c+d-e-f \end{smallmatrix}\right] \\ &= au_1 + bu_2 + cu_3 + du_4 + (e-a-c)v_1 + (f-b-d)v_2. \end{aligned}$$

Using this, we see that $u_1, \dots, u_4, v_1, v_2$ is a basis for V. Similarly, $u_1, \dots, u_4, w_1, w_2$ is a basis for W. It follows that

$$u_1, u_2, u_3, u_4, v_1, v_2, w_1, w_2$$

is a basis for $V + W$.

Example 8.16. Put $U = \mathbb{R}[x]_{\leq 3}$ and

$$\begin{aligned} V &= \{f \in U \mid f(1) = 0\} = \{(x-1)g(x) \mid g(x) \in \mathbb{R}[x]_{\leq 2}\} \\ W &= \{f \in U \mid f(-1) = 0\} = \{(x+1)h(x) \mid h(x) \in \mathbb{R}[x]_{\leq 2}\} \\ \text{so } V \cap W &= \{f \in U \mid f \text{ is divisible by } (x+1)(x-1) = x^2 - 1\} \end{aligned}$$

Any $f(x) \in V \cap W$ has the form $(ax+b)(x^2-1) = a(x^3-x) + b(x^2-1)$. It follows that the list $u_1 = x^3 - x, u_2 = x^2 - 1$ is a basis for $V \cap W$. Now put $v_1 = x - 1 \in V$ and $w_1 = x + 1 \in W$. We claim that u_1, u_2, v_1 is a basis for V. Indeed, any element of V has the form

$$f(x) = (ax^2 + bx + c).(x-1) = (ax^2 + (b-c)x + c(x+1)).(x-1) = au_1 + (b-c)u_2 + cv_1,$$

so the list spans V. If we have a linear relation $au_1 + bu_2 + cv_1 = 0$ then $a(x^3 - x) + b(x^2 - 1) + c(x-1) = 0$ for all x, so $ax^3 + bx^2 + (c-a)x - c = 0$ for all x, which implies that $a = b = c = 0$. Our list is thus independent as well as spanning V, so it is a basis. Similarly u_1, u_2, w_1 is a basis for W. It follows that u_1, u_2, v_1, w_1 is a basis for $V + W$.

Theorem 8.17. *Let $\alpha \colon U \to V$ be a linear map between finite-dimensional vector spaces. Then one can choose a basis $\mathcal{U} = u_1, \dots, u_m$ for U, and a basis $\mathcal{V} = v_1, \dots, v_n$ for V, and an integer $r \leq \min(m,n)$ such that*

(a) $\alpha(u_i) = v_i$ *for* $1 \leq i \leq r$
(b) $\alpha(u_i) = 0$ *for* $r < i \leq m$
(c) $u_{r+1}, \dots, u_m$ *is a basis for* $\ker(\alpha) \leq U$
(d) $v_1, \dots, v_r$ *is a basis for* $\operatorname{image}(\alpha) \leq V$.

Proof. Let $v_1, \dots, v_r$ be any basis for $\operatorname{image}(\alpha)$ (so (d) is satisfied). By Proposition 8.11, this can be extended to a list $\mathcal{V} = v_1, \dots, v_n$ which is a basis for all of V. Next, for $j \leq r$ we have $v_j \in \operatorname{image}(\alpha)$, so we can choose $u_j \in U$ with $\alpha(u_j) = v_j$ (so (a) is satisfied). This gives us a list $u_1, \dots, u_r$ of elements of U; to these, we add vectors $u_{r+1}, \dots, u_m$ forming a basis for $\ker(\alpha)$ (so that (b) and (c) are satisfied). Now everything is as claimed except that we have not shown that the list $\mathcal{U} = u_1, \dots, u_n$ is a basis for U.

Consider an element $x \in U$. We then have $\alpha(x) \in \operatorname{image}(\alpha)$, and $v_1, \dots, v_r$ is a basis for $\operatorname{image}(\alpha)$, so there exist numbers $x_1, \dots, x_r$ such that $\alpha(x) = x_1 v_1 + \ldots + x_r v_r$. Now put $x' = x_1 u_1 + \ldots + x_r u_r$, and $x'' = x - x'$. We have

$$\alpha(x') = x_1 \alpha(u_1) + \cdots + x_r \alpha(u_r) = x_1 v_1 + \cdots + x_r v_r = \alpha(x),$$

so $\alpha(x'') = \alpha(x) - \alpha(x') = 0$, so $x'' \in \ker(\alpha)$. We also know that $u_{r+1}, \dots, u_m$ is a basis for $\ker(\alpha)$, so there exist numbers $x_{r+1}, \dots, x_m$ with $x'' = x_{r+1} u_{r+1} + \cdots + x_m u_m$. Putting this together, we get

$$x = x' + x'' = (x_1 u_1 + \cdots + x_r u_r) + (x_{r+1} u_{r+1} + \cdots + x_m u_m),$$

which is a linear combination of $u_1, \dots, u_m$. It follows that the list $\mathcal{U}$ spans U.

Now suppose we have a linear relation $\lambda_1 u_1 + \cdots + \lambda_m u_m = 0$. We apply α to both sides of this equation to get

$$\begin{aligned} 0 &= \lambda_1 \alpha(u_1) + \cdots + \lambda_r \alpha(u_r) + \lambda_{r+1} \alpha(u_{r+1}) + \cdots + \lambda_m \alpha(u_m) \\ &= \lambda_1 v_1 + \cdots + \lambda_r v_r + \lambda_{r+1}.0 + \cdots + \lambda_m.0 \\ &= \lambda_1 v_1 + \cdots + \lambda_r v_r. \end{aligned}$$

This is a linear relation between the vectors $v_1, \dots, v_r$, but these form a basis for $\operatorname{image}(\alpha)$, so this must be the trivial relation, so $\lambda_1 = \cdots = \lambda_r = 0$. This means that our original relation has the form

$$\lambda_{r+1} u_{r+1} + \cdots + \lambda_m u_m = 0$$

As $u_{r+1}, \dots, u_m$ is a basis for $\ker(\alpha)$, these vectors are linearly independent, so the above relation must be trivial, so $\lambda_{r+1} = \cdots = \lambda_m = 0$. This shows that all the λ's are zero, so the original relation was trivial. Thus, the vectors $u_1, \dots, u_m$ are linearly independent, as claimed. □

Remark 8.18. If we use bases as in the theorem, then the matrix of α with respect to those bases has the form

$$A = \left[\begin{array}{c|c} I_r & 0_{r,m-r} \\ 0_{n-r,r} & 0_{n-r,m-r} \end{array}\right]$$

Corollary 8.19. *If $\alpha\colon U \to V$ is a linear map then*

$$\dim(\ker(\alpha)) + \dim(\mathrm{image}(\alpha)) = \dim(U).$$

Proof. Choose bases as in the theorem. Then $\dim(U) = m$ and $\dim(\mathrm{image}(\alpha)) = r$ and

$$\dim(\ker(\alpha)) = |\{u_{r+1}, \dots, u_m\}| = m - r.$$

The claim follows. □

Example 8.20. Consider the map $\phi\colon M_2\mathbb{R} \to M_2\mathbb{R}$ given by $\phi(A) = \left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right] A \left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]$, or equivalently

$$\begin{aligned}\phi\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right] &= \left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right]\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right] = \left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]\left[\begin{smallmatrix}b&a\\d&c\end{smallmatrix}\right] = \left[\begin{smallmatrix}b+d&a+c\\b+d&a+c\end{smallmatrix}\right] \\ &= (b+d)\left[\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right] + (a+c)\left[\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\right].\end{aligned}$$

It follows that if we put

$$u_1 = \left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right] \quad u_2 = \left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right] \quad v_1 = \left[\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right] \quad v_2 = \left[\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\right]$$

then $\phi(u_1) = v_1$, $\phi(u_2) = v_2$, and v_1, v_2 is a basis for $\mathrm{image}(\phi)$. It can be extended to a basis for all of $M_2\mathbb{R}$ by adding $v_3 = \left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right]$ and $v_4 = \left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right]$. Moreover, we have $\phi(A) = 0$ iff $a + c = b + d = 0$ iff $c = -a$ and $d = -b$, in which case

$$A = \left[\begin{smallmatrix}a&b\\-a&-b\end{smallmatrix}\right] = a\left[\begin{smallmatrix}1&0\\-1&0\end{smallmatrix}\right] + b\left[\begin{smallmatrix}0&1\\0&-1\end{smallmatrix}\right].$$

This means that the matrices $u_3 = \left[\begin{smallmatrix}1&0\\-1&0\end{smallmatrix}\right]$ and $u_4 = \left[\begin{smallmatrix}0&1\\0&-1\end{smallmatrix}\right]$ form a basis for $\ker(\phi)$. Putting this together, we see that $u_1, \dots, u_4$ and $v_1, \dots, v_4$ are bases for $M_2\mathbb{R}$ such that $\phi(u_i) = v_i$ for $i \leq 2$, and $\phi(u_i) = 0$ for $i > 2$.

9. Eigenvalues and eigenvectors

Definition 9.1. Let V be a finite-dimensional vector space over $\mathbb{C}$, and let $\alpha\colon V \to V$ be a $\mathbb{C}$-linear map. Let λ be a complex number. An *eigenvector* for α, with *eigenvalue* λ is a nonzero element $v \in V$ such that $\alpha(v) = \lambda v$. If such a v exists, we say that λ is an *eigenvalue of* α.

Remark 9.2. Suppose we choose a basis $\mathcal{V}$ for V, and let A be the matrix of α with respect to $\mathcal{V}$ and $\mathcal{V}$. Then the eigenvalues of α are the same as the eigenvalues of the matrix A, which are the roots of the characteristic polynomial $\det(tI - A)$.

Example 9.3. Put $V = \mathbb{C}[x]_{\leq 4}$, and define $\phi\colon V \to V$ by $\phi(f)(x) = f(x+1)$. We claim that 1 is the only eigenvalue. Indeed, the corresponding matrix P (with respect to the basis $1, x, \dots, x^4$) is

$$P = \left[\begin{smallmatrix}1&1&1&1&1\\0&1&2&3&4\\0&0&1&3&6\\0&0&0&1&4\\0&0&0&0&1\end{smallmatrix}\right]$$

The characteristic polynomial is thus

$$\det(tI - P) = \det\left[\begin{smallmatrix}t-1&-1&-1&-1&-1\\0&t-1&-2&-3&-4\\0&0&t-1&-3&-6\\0&0&0&t-1&-4\\0&0&0&0&t-1\end{smallmatrix}\right] = (t-1)^5$$

so 1 is the only root of the characteristic polynomial. The eigenvectors are just the polynomials f with $\phi(f) = 1.f$ or equivalently $f(x+1) = f(x)$ for all x. These are just the constant polynomials.

Example 9.4. Put $V = \mathbb{C}[x]_{\leq 4}$, and define $\phi\colon V \to V$ by $\phi(f)(x) = f(ix)$, so $\phi(x^k) = i^k x^k$. The corresponding matrix P (with respect to $1, x, x^2, x^3, x^4$) is

$$P = \left[\begin{smallmatrix}1&0&0&0&0\\0&i&0&0&0\\0&0&-1&0&0\\0&0&0&-i&0\\0&0&0&0&1\end{smallmatrix}\right]$$

The characteristic polynomial is thus

$$\det(tI - P) = (t-1)(t-i)(t+1)(t+i)(t-1) = (t-1)(t^2+1)(t^2-1) = t^5 - t^4 - t + 1$$

so the eigenvalues are $1, i, -1$ and $-i$.

- The eigenvectors of eigenvalue 1 are functions $f \in V$ with $f(ix) = f(x)$. These are the functions of the form $f(x) = a + ex^4$.
- The eigenvectors of eigenvalue i are functions $f \in V$ with $f(ix) = if(x)$. These are the functions of the form $f(x) = bx$.
- The eigenvectors of eigenvalue -1 are functions $f \in V$ with $f(ix) = -f(x)$. These are the functions of the form $f(x) = cx^2$.
- The eigenvectors of eigenvalue $-i$ are functions $f \in V$ with $f(ix) = -if(x)$. These are the functions of the form $f(x) = dx^3$.

Example 9.5. Let $\mathbf{u}$ be a unit vector in $\mathbb{R}^3$ and define $\alpha\colon \mathbb{C}^3 \to \mathbb{C}^3$ by $\alpha(\mathbf{v}) = \mathbf{u}\times\mathbf{v}$. Choose a unit vector $\mathbf{b}$ orthogonal to $\mathbf{a}$ and put $\mathbf{c} = \mathbf{a}\times\mathbf{b}$. We saw previously that the matrix of α with respect to $\mathbf{a},\mathbf{b},\mathbf{c}$ is

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

The characteristic polynomial is thus

$$\det(tI - A) = \det\begin{bmatrix} t & 0 & 0 \\ 0 & t & 1 \\ 0 & -1 & t \end{bmatrix} = t^3 + t = t(t+i)(t-i).$$

The eigenvalues are thus 0, i and $-i$.

- The eigenvectors of eigenvalue 0 are the multiples of $\mathbf{a}$.
- The eigenvectors of eigenvalue i are the multiples of $\mathbf{b} - i\mathbf{c}$.
- The eigenvectors of eigenvalue $-i$ are the multiples of $\mathbf{b} + i\mathbf{c}$.

Example 9.6. Let $\mathbf{u}$ and $\mathbf{v}$ be non-orthogonal vectors in $\mathbb{R}^3$, and define $\phi\colon \mathbb{C}^3 \to \mathbb{C}^3$ by $\phi(\mathbf{x}) = \langle \mathbf{u},\mathbf{x}\rangle\mathbf{v}$. We claim that the characteristic polynomial of ϕ is $t^2(t - \mathbf{u}.\mathbf{v})$. Indeed, the matrix P with respect to the standard basis is calculated as follows:

$$\phi(\mathbf{e}_1) = u_1\mathbf{v} = \begin{bmatrix} u_1v_1 \\ u_1v_2 \\ u_1v_3 \end{bmatrix} \qquad \phi(\mathbf{e}_2) = u_2\mathbf{v} = \begin{bmatrix} u_2v_1 \\ u_2v_2 \\ u_2v_3 \end{bmatrix} \qquad \phi(\mathbf{e}_3) = u_3\mathbf{v} = \begin{bmatrix} u_3v_1 \\ u_3v_2 \\ u_3v_3 \end{bmatrix}$$

$$P = \begin{bmatrix} u_1v_1 & u_2v_1 & u_3v_1 \\ u_1v_2 & u_2v_2 & u_3v_2 \\ u_1v_3 & u_2v_3 & u_3v_3 \end{bmatrix}$$

The characteristic polynomial is $\det(tI - P) = -\det(P - tI)$, which is found as follows:

$$\begin{aligned} &\det(P - tI) \\ =&\det\begin{bmatrix} u_1v_1-t & u_2v_1 & u_3v_1 \\ u_1v_2 & u_2v_2-t & u_3v_2 \\ u_1v_3 & u_2v_3 & u_3v_3-t \end{bmatrix} \\ =&(u_1v_1 - t)\det\begin{bmatrix} u_2v_2-t & u_3v_2 \\ u_2v_3 & u_3v_3-t \end{bmatrix} - u_2v_1\det\begin{bmatrix} u_1v_2 & u_3v_2 \\ u_1v_3 & u_3v_3-t \end{bmatrix} + u_3v_1\det\begin{bmatrix} u_1v_2 & u_2v_2-t \\ u_1v_3 & u_2v_3 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \det\begin{bmatrix} u_2v_2-t & u_3v_2 \\ u_2v_3 & u_3v_3-t \end{bmatrix} &= (u_2v_2 - t)(u_3v_3 - t) - u_2v_3u_3v_2 = t^2 - (u_2v_2 + u_3v_3)t \\ \det\begin{bmatrix} u_1v_2 & u_3v_2 \\ u_1v_3 & u_3v_3-t \end{bmatrix} &= u_1v_2(u_3v_3 - t) - u_1v_3u_3v_2 = -u_1v_2t \\ \det\begin{bmatrix} u_1v_2 & u_2v_2-t \\ u_1v_3 & u_2v_3 \end{bmatrix} &= u_1v_2u_2v_3 - u_1v_3(u_2v_2 - t) = u_1v_3t \\ \det(P - tI) &= (u_1v_1 - t)(t^2 - (u_2v_2 + u_3v_3)t) - u_2v_1(-u_1v_2t) + u_3v_1u_1v_3t \\ &= (u_1v_1 + u_2v_2 + u_3v_3)t^2 - t^3 \\ \det(tI - P) &= t^3 - (u_1v_1 + u_2v_2 + u_3v_3)t^2 = t^2(t - \langle \mathbf{u},\mathbf{v}\rangle) \end{aligned}$$

The eigenvalues are thus 0 and $\langle \mathbf{u},\mathbf{v}\rangle$. The eigenvectors of eigenvalue 0 are the vectors orthogonal to $\mathbf{u}$. The eigenvectors of eigenvalue $\langle \mathbf{u},\mathbf{v}\rangle$ are the multiples of $\mathbf{v}$.

If we had noticed this in advance then the whole argument would have been much easier. We could have chosen a basis of the form $\mathbf{a},\mathbf{b},\mathbf{v}$ with $\mathbf{a}$ and $\mathbf{b}$ orthogonal to $\mathbf{u}$. With respect to that basis, ϕ would have matrix $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \langle \mathbf{u},\mathbf{v}\rangle \end{bmatrix}$ which immediately gives the characteristic polynomial.

10. Inner products

Definition 10.1. Let V be a vector space over $\mathbb{R}$. An *inner product* on V is a rule that gives a number $\langle u, v\rangle \in \mathbb{R}$ for each $u, v \in V$, with the following properties:

(a) $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$ for all $u, v, w \in V$.
(b) $\langle tu, v\rangle = t\langle u, v\rangle$ for all $u, v \in V$ and $t \in \mathbb{R}$.
(c) $\langle u, v\rangle = \langle v, u\rangle$ for all $u, v \in V$.
(d) We have $\langle u, u\rangle \geq 0$ for all $u \in V$, and $\langle u, u\rangle = 0$ iff $u = 0$.

Given an inner product, we will write $\|u\| = \sqrt{\langle u, u\rangle}$, and call this the *norm* of u. We say that u is a *unit vector* if $\|u\| = 1$.

Remark 10.2. Unlike most of the other things we have done, this does not immediately generalise to fields K other than $\mathbb{R}$. The reason is that axiom (d) involves the condition $\langle u, u\rangle \geq 0$, and in an arbitrary field K (such as $\mathbb{Z}/5$, for example) we do not have a good notion of positivity. Moreover, all our examples will rely heavily on the fact that $x^2 \geq 0$ for all $x \in \mathbb{R}$, and of course this ceases to be true if we work over $\mathbb{C}$. We will see in Section 13 how to fix things up in the complex case.

Example 10.3. We can define an inner product on $\mathbb{R}^n$ by

$$\langle (x_1,\dots,x_n),(y_1,\dots,y_n)\rangle = \sum_{i=1}^n x_iy_i = x_1y_1 + x_2y_2 + \cdots + x_ny_n.$$

Properties (a) to (c) are obvious. For property (d), note that if $\mathbf{u} = [u_1, \dots, u_n]^T \in \mathbb{R}^n$ then

$$\langle \mathbf{u}, \mathbf{u} \rangle = u_1^2 + u_2^2 + \cdots + u_n^2.$$

All the terms in this sum are at least zero, so the sum must be at least zero. Moreover, there can be no cancellation, so the only way that $\langle \mathbf{u}, \mathbf{u} \rangle$ can be zero is if all the individual terms are zero, which means $u_1 = u_2 = \cdots = u_n = 0$, so $\mathbf{u} = 0$ as a vector.

Remark 10.4. If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ then we can regard $\mathbf{x}$ and $\mathbf{y}$ as $n \times 1$ matrices, so $\mathbf{x}^T$ is a $1 \times n$ matrix, so $\mathbf{x}^T\mathbf{y}$ is a 1×1 matrix, or in other words a number. This number is just $\langle \mathbf{x}, \mathbf{y} \rangle$. This is most easily explained by example: in the case $n = 4$ we have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}^T \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 = \langle \mathbf{x}, \mathbf{y} \rangle.$$

Example 10.5. Although we usually use the standard inner product on $\mathbb{R}^n$, there are many other inner products. For example, we can define a different inner product on $\mathbb{R}^3$ by

$$\langle (u, v, w), (x, y, z) \rangle' = ux + (u + v)(x + y) + (u + v + w)(x + y + z).$$

In particular, this gives

$$\langle (u, v, w), (u, v, w) \rangle' = u^2 + (u + v)^2 + (u + v + w)^2 = 3u^2 + 2v^2 + w^2 + 4uv + 2vw + 2uw.$$

The corresponding norm is thus

$$\|(u, v, w)\|' = \sqrt{\langle (u, v, w), (u, v, w) \rangle'} = \sqrt{3u^2 + 2v^2 + w^2 + 4uv + 2vw + 2uw}.$$

Example 10.6. Let U be the set of physical vectors, as in Example 2.6. Given $\mathbf{u}, \mathbf{v} \in U$ we can define

$$\langle \mathbf{u}, \mathbf{v} \rangle = (\text{length of } \mathbf{u} \text{ in miles}) \times (\text{length of } \mathbf{v} \text{ in miles}) \times \cos(\text{ angle between } \mathbf{u} \text{ and } \mathbf{v}\ ).$$

This turns out to give an inner product on U. Of course we could use a different unit of length instead of miles, and that would just change the inner product by a constant factor.

Example 10.7. We can define an inner product on $C[0, 1]$ by

$$\langle f, g \rangle = \int_{x=0}^{1} f(x)g(x)\, dx.$$

Properties (a) to (c) are obvious. For property (d), note that if $f \in C[0, 1]$ then

$$\langle f, f \rangle = \int_0^1 f(x)^2\, dx$$

As $f(x)^2 \geq 0$ for all x, we have $\langle f, f \rangle \geq 0$. If $\langle f, f \rangle = 0$ then the area between the x-axis and the graph of $f(x)^2$ is zero, so $f(x)^2$ must be zero for all x, so $f = 0$ as required.

Here is a slightly more careful version of the argument. Suppose that f is nonzero. We can then find some number a with $0 \leq a \leq 1$ and $f(a) > 0$. Put $\epsilon = f(a)/2$. As f is continuous, there exists $\delta > 0$ such that whenever $x \in [0, 1]$ and $|x - a| < \delta$ we have $|f(x) - f(a)| < \epsilon$. For such x we have

$$f(a) - \epsilon < f(x) < f(a) + \epsilon,$$

but $f(a) = 2\epsilon$ so $f(x) > \epsilon$. Usually we will be able to say that f is greater than ϵ on the interval $(a - \delta, a + \delta)$ which has length $2\delta > 0$, so

$$\int_0^1 f(x)^2\, dx \geq \int_{a-\delta}^{a+\delta} \epsilon^2\, dx = 2\delta\epsilon^2 > 0.$$

This will not be quite right, however, if $a - \delta < 0$ or $a + \delta > 1$, because then the interval $(a - \delta, a + \delta)$ is not contained in the domain where f is defined. However, we can still put $a_- = max(a - \delta, 0)$ and $a_+ = \min(a + \delta, 1)$ and we find that $a_- < a_+$ and

$$\langle f, f \rangle = \int_0^1 f(x)^2\, dx \geq \int_{a_-}^{a_+} \epsilon^2\, dx = (a_+ - a_-)\epsilon^2 > 0,$$

as required.

Example 10.8. We can define an inner product on the space $M_n\mathbb{R}$ by

$$\langle A, B \rangle = \operatorname{trace}(AB^T)$$

Consider for example the case $n = 3$, so

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

so

$$AB^T = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \\ b_{13} & b_{23} & b_{33} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11}+a_{12}b_{12}+a_{13}b_{13} & a_{11}b_{21}+a_{12}b_{22}+a_{13}b_{23} & a_{11}b_{31}+a_{12}b_{32}+a_{13}b_{33} \\ a_{21}b_{11}+a_{22}b_{12}+a_{23}b_{13} & a_{21}b_{21}+a_{22}b_{22}+a_{23}b_{23} & a_{21}b_{31}+a_{22}b_{32}+a_{23}b_{33} \\ a_{31}b_{11}+a_{32}b_{12}+a_{33}b_{13} & a_{31}b_{21}+a_{32}b_{22}+a_{33}b_{23} & a_{31}b_{31}+a_{32}b_{32}+a_{33}b_{33} \end{bmatrix}$$

so

$$\begin{aligned} \langle A, B\rangle = \text{trace}(AB^T) =& a_{11}b_{11} + a_{12}b_{12} + a_{13}b_{13} + \\ & a_{21}b_{21} + a_{22}b_{22} + a_{23}b_{23} + \\ & a_{31}b_{31} + a_{32}b_{32} + a_{33}b_{33} \\ =& \sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}b_{ij}. \end{aligned}$$

In other words $\langle A, B\rangle$ is the sum of the entries of A multiplied by the corresponding entries in B. Thus, if we identify $M_3\mathbb{R}$ with $\mathbb{R}^9$ in the usual way, then our inner product on $M_3\mathbb{R}$ corresponds to the standard inner product on $\mathbb{R}^9$. Similarly, if we identify $M_n\mathbb{R}$ with $\mathbb{R}^{n^2}$ in the usual way, then our inner product on $M_n\mathbb{R}$ corresponds to the standard inner product on $\mathbb{R}^{n^2}$.

Example 10.9. For any $a < b$ we can define an inner product $\langle ;\rangle_{[a,b]}$ on $\mathbb{R}[x]_{\leq 2}$ by

$$\langle u, v\rangle_{[a,b]} = \int_a^b u(x)v(x)\,dx.$$

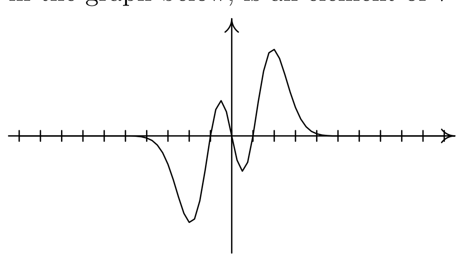
In particular, we have

$$\langle x^i, x^j\rangle_{[a,b]} = \int_a^b x^{i+j}\,dx = \left[\frac{x^{i+j+1}}{i+j+1}\right]_a^b = \frac{b^{i+j+1} - a^{i+j+1}}{i+j+1}$$

This gives an infinite family of different inner products on $\mathbb{R}[x]_{\leq 2}$. For example:

$$\begin{aligned} \langle 1, x^2\rangle_{[-1,1]} &= \frac{1^3 - (-1)^3}{3} = 2/3 \\ \langle x, x^2\rangle_{[-1,1]} &= \frac{1^4 - (-1)^4}{4} = 0 \\ \|x^2\|_{[-1,1]} &= \sqrt{\frac{1^5 - (-1)^5}{5}} = \sqrt{2/5} \\ \|x^2\|_{[0,5]} &= \sqrt{\frac{5^5 - 0^5}{5}} = 25 \end{aligned}$$

Example 10.10. Let V be the set of functions of the form $p(x)e^{-x^2/2}$, where $p(x)$ is a polynomial. For example, the function $f(x) = (x^3 - x)e^{-x^2/2}$, shown in the graph below, is an element of V:



We can define an inner product on V by

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)g(x)\,dx$$

Note that this only works because of the special form of the functions in V. For most functions f and g that you might think of, the integral $\int_{-\infty}^{\infty} f(x)g(x)\,dx$ will give an infinite or undefined answer. However, the function e^{-x^2} decays very rapidly to zero as $|x|$ tends to infinity, and one can check that this is enough to make the integral well-defined and finite when f and g are in V. In fact, we have the formula

$$\begin{aligned} \langle x^n e^{-x^2/2}, x^m e^{-x^2/2}\rangle &= \int_{-\infty}^{\infty} x^{n+m} e^{-x^2}\,dx \\ &= \begin{cases} \frac{\sqrt{\pi}}{2^{n+m}} \frac{(n+m)!}{((n+m)/2)!} & \text{if } n+m \text{ is even} \\ 0 & \text{if } n+m \text{ is odd} \end{cases} \end{aligned}$$

11. The Cauchy-Schwartz inequality

If $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$, you should be familiar with the fact that

$$\langle \mathbf{v}, \mathbf{w} \rangle = \|\mathbf{v}\| \, \|\mathbf{w}\| \, \cos(\theta),$$

where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$. In particular, as the cosine lies between -1 and 1, we see that $|\langle \mathbf{v}, \mathbf{w} \rangle| \leq \|\mathbf{v}\| \, \|\mathbf{w}\|$. We would like to extend all this to arbitrary inner-product spaces.

Theorem 11.1 (The Cauchy-Schwartz inequality). *Let V be an inner product space over $\mathbb{R}$, and let v and w be elements of V. Then*

$$|\langle v, w \rangle| \leq \|v\| \, \|w\|,$$

with equality iff v and w are linearly dependent.

Proof. If $w = 0$ then $|\langle v, w \rangle| = 0 = \|v\| \, \|w\|$ and v and w are linearly dependent, so the theorem holds. For the rest of the proof, we can thus restrict attention to the other case, where $w \neq 0$.

For any real numbers s and t, we have

$$0 \leq \|sv + tw\|^2 = \langle sv + tw, sv + tw \rangle = s^2 \langle v, v \rangle + st\langle v, w \rangle + st\langle w, v \rangle + t^2 \langle w, w \rangle = s^2 \|v\|^2 + 2st\langle v, w \rangle + t^2 \|w\|^2.$$

Now take $s = \langle w, w \rangle = \|w\|^2$ and $t = -\langle v, w \rangle$. The above inequality gives

$$\begin{aligned} 0 &\leq \|w\|^4 \|v\|^2 - 2\|w\|^2 \langle v, w \rangle^2 + \langle v, w \rangle^2 \|w\|^2 \\ &= \|w\|^2 (\|w\|^2 \|v\|^2 - \langle v, w \rangle^2). \end{aligned}$$

We have assumed that $w \neq 0$, so $\|w\|^2 > 0$. We can thus divide by $\|w\|^2$ and rearrange to see that $\langle v, w \rangle^2 \leq \|v\|^2 \|w\|^2$. It follows that $|\langle v, w \rangle| \leq \|v\| \|w\|$ as claimed.

If we have equality (i.e. $|\langle v, w \rangle| = \|v\| \|w\|$) then our calculation shows that $\|sv + tw\|^2 = 0$, so $sv + tw = 0$. Here $s = \|w\|^2 > 0$, so we have a *nontrivial* linear relation between v and w, so they are linearly dependent.

Conversely, suppose we start by assuming that v and w are linearly dependent. As $w \neq 0$, this means that $v = \lambda w$ for some $\lambda \in \mathbb{R}$. It follows that $\langle v, w \rangle = \lambda \|w\|^2$, so $|\langle v, w \rangle\| = |\lambda| \|w\|^2$. On the other hand, we have $\|v\| = |\lambda| \|w\|$, so $\|v\| \, \|w\| = |\lambda| \|w\|^2$, which is the same. □

Example 11.2. We claim that for any vector $\mathbf{x} \in \mathbb{R}^n$, we have

$$|x_1 + \cdots + x_n| \leq \sqrt{n} \sqrt{x_1^2 + \cdots + x_n^2}.$$

To see this, use the standard inner product on $\mathbb{R}^n$, and consider the vector $\mathbf{e} = [1, 1, \ldots, 1]^T$. We have

$$\begin{aligned} \|\mathbf{x}\| &= \sqrt{x_1^2 + \cdots + x_n^2} \\ \|\mathbf{e}\| &= \sqrt{n} \\ \langle \mathbf{x}, \mathbf{e} \rangle &= x_1 + \cdots + x_n. \end{aligned}$$

The Cauchy-Schwartz inequality therefore tells us that

$$\begin{aligned} |x_1 + \cdots + x_n| &= |\langle \mathbf{x}, \mathbf{e} \rangle| \\ &\leq \|\mathbf{x}\| \|\mathbf{e}\| = \sqrt{x_1^2 + \cdots + x_n^2} \sqrt{n}, \end{aligned}$$

as claimed.

Example 11.3. We claim that for any continuous function $f \colon [0, 1] \to \mathbb{R}$ we have

$$\left| \int_0^1 (1 - x^2) f(x) \, dx \right| \leq \sqrt{\frac{8}{15}} \sqrt{\int_0^1 f(x)^2 \, dx}.$$

Indeed, we can define an inner product on $C[0, 1]$ by $\langle u, v \rangle = \int_0^1 u(x)v(x) \, dx$. We then have $\|f\| = \sqrt{\int_0^1 f(x)^2 \, dx}$ and

$$\begin{aligned} \|1 - x^2\|^2 &= \langle 1 - x^2, 1 - x^2 \rangle = \int_0^1 1 - 2x^2 + x^4 \, dx \\ &= \left[x - \tfrac{2}{3}x^3 + \tfrac{1}{5}x^5\right]_0^1 = 1 - \tfrac{2}{3} + \tfrac{1}{5} = 8/15 \\ \|1 - x^2\| &= \sqrt{8/15} \end{aligned}$$

The Cauchy-Schwartz inequality tells us that $|\langle u, f \rangle| \leq \|u\| \, \|f\|$, so $\left| \int_0^1 (1 - x^2) f(x) \, dx \right| \leq \sqrt{\frac{8}{15}} \sqrt{\int_0^1 f(x)^2 \, dx}$. as claimed.

Example 11.4. Let A be a nonzero $n \times n$ matrix over $\mathbb{R}$. We claim that

(a) $\operatorname{trace}(A)^2 \leq n \operatorname{trace}(AA^T)$, with equality iff A is a multiple of the identity.
(b) $\operatorname{trace}(A^2) \leq \operatorname{trace}(AA^T)$, with equality iff A is either symmetric or antisymmetric.

In both cases we use the inner product $\langle A, B\rangle = \operatorname{trace}(AB^T)$ on $M_n\mathbb{R}$ and the Cauchy-Schwartz inequality.

(a) Apply the inequality to A and I, giving $|\langle A, I\rangle| \leq \|A\|\|I\|$, or equivalently
$$\langle A, I\rangle^2 \leq \|A\|^2\|I\|^2 = \operatorname{trace}(AA^T)\operatorname{trace}(II^T)$$
Here $\langle A, I\rangle = \operatorname{trace}(A)$ and $\operatorname{trace}(II^T) = \operatorname{trace}(I) = n$, so we get $\operatorname{trace}(A)^2 \leq n \operatorname{trace}(AA^T)$ as claimed. This is an equality iff A and I are linearly dependent, which means that A is a multiple of I.

(b) Now instead apply the inequality to A and A^T, noting that $\|A\| = \|A^T\| = \sqrt{\operatorname{trace}(AA^T)}$ and $\langle A, A^T\rangle = \operatorname{trace}(AA^{TT}) = \operatorname{trace}(A^2)$. The conclusion is that $|\operatorname{trace}(A^2)| \leq \sqrt{\operatorname{trace}(AA^T)}\sqrt{\operatorname{trace}(AA^T)}$, which gives $\operatorname{trace}(A^2) \leq \operatorname{trace}(AA^T)$. This is an equality iff A^T is a multiple of A, say $A^T = \lambda A$ for some λ. This means that $A = A^{TT} = \lambda A^T = \lambda^2 A$, and $A \neq 0$, so this means that $\lambda^2 = 1$, or equivalently $\lambda = \pm 1$. If $\lambda = 1$ then $A^T = A$ and A is symmetric; if $\lambda = -1$ then $A^T = -A$ and A is antisymmetric.

It is now natural to ask whether we also have $\langle v, w\rangle = \|v\|\,\|w\|\,\cos(\theta)$ (where θ is the angle between v and w), just as we did in $\mathbb{R}^3$. However, the question is meaningless as it stands, because we do not yet have a definition of angles between elements of an arbitrary inner-product space. We will use the following definition, which makes the above equation true by tautology.

Definition 11.5. Let V be an inner product space over $\mathbb{R}$, and let v and w be nonzero elements of V, so $\|v\|\,\|w\| > 0$. Put $c = \langle v, w\rangle/(\|v\|\,\|w\|)$. The Cauchy-Schwartz inequality tells us that $-1 \leq c \leq 1$, so there is a unique angle $\theta \in [0, \pi]$ such that $\cos(\theta) = c$. We call this the angle between v and w.

Example 11.6. Take $V = C[0, 1]$ (with the usual inner product), and $v(t) = 1$, and $w(t) = t$. Then $\|v\| = 1$ and $\|w\| = 1/\sqrt{3}$ and $\langle v, w\rangle = 1/2$, so $\langle v, w\rangle/(\|v\|\,\|w\|) = \sqrt{3}/2 = \cos(\pi/6)$, so the angle between v and w is $\pi/6$.

Example 11.7. Take $V = M_3\mathbb{R}$ (with the usual inner product) and
$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$
We then have
$$\|A\| = \sqrt{0^2 + 1^2 + 0^2 + 1^2 + 2^2 + 1^2 + 0^2 + 1^2 + 0^2} = \sqrt{8} = 2\sqrt{2}$$
$$\|B\| = \sqrt{1^2 + 0^2 + 0^2 + 1^2 + 1^2 + 1^2 + 0^2 + 0^2 + 0^2} = \sqrt{4} = 2$$
$$\langle A, B\rangle = 0.1 + 1.0 + 0.0 + 1.1 + 2.1 + 1.1 + 0.0 + 1.0 + 0.0 = 4$$
so $\langle A, B\rangle/(\|A\|\|B\|) = 4/(4\sqrt{2}) = 1/\sqrt{2} = \cos(\pi/4)$. The angle between A and B is thus $\pi/4$.

12. Projections and the Gram-Schmidt procedure

Definition 12.1. Let V be a vector space with inner product, and let W be a subspace. We then put
$$W^\perp = \{v \in V \mid \langle v, w\rangle = 0 \text{ for all } w \in W\}.$$
This is called the *orthogonal complement* (or *annihilator*) of W. We say that W is *complemented* if $W + W^\perp = V$.

Lemma 12.2. *We always have $W \cap W^\perp = 0$. (Thus, if W is complemented, we have $V = W \oplus W^\perp$.)*

Proof. Suppose that $v \in W \cap W^\perp$. As $v \in W^\perp$, we have $\langle v, w\rangle = 0$ for all $w \in W$. As $v \in W$, we can take $w = v$, which gives $\|v\|^2 = \langle v, v\rangle = 0$. This implies that $v = 0$, as required. □

Definition 12.3. Let V be a vector space with inner product. We say that a sequence $\mathcal{V} = v_1, \ldots, v_n$ of elements of V is *orthogonal* if we have $\langle v_i, v_j\rangle = 0$ for all $i \neq j$. We say that the sequence is *strictly orthogonal* if it is orthogonal, and all the elements v_i are nonzero. We say that the sequence is *orthonormal* if it is orthogonal, and also $\langle v_i, v_i\rangle = 1$ for all i.

Remark 12.4. If $\mathcal{V}$ is a strictly orthogonal sequence then we can define an orthonormal sequence $\hat{v}_1, \ldots, \hat{v}_n$ by $\hat{v}_i = v_i/\|v_i\|$.

Example 12.5. The standard basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$ for $\mathbb{R}^n$ is an orthonormal sequence.

Example 12.6. Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be the vectors joining the centre of the earth to the North Pole, the mouth of the river Amazon, and the city of Mogadishu. These are elements of the inner product space U discussed in Examples 2.6 and 10.6. Then $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is an orthogonal sequence, and $\mathbf{a}/4000, \mathbf{b}/4000, \mathbf{c}/4000$ is an orthonormal sequence. (Of course, these statements are only approximations. You can take it as an internet exercise to work out the size of the errors involved.)

Lemma 12.7. *Let $v_1,\dots,v_n$ be an orthogonal sequence, and put $v = v_1 + \dots + v_n$. Then*

$$\|v\| = \sqrt{\|v_1\|^2 + \dots + \|v_n\|^2}.$$

Proof. We have

$$\|v\|^2 = \langle \sum_i v_i, \sum_j v_j \rangle = \sum_{i,j} \langle v_i, v_j \rangle.$$

Because the sequence is orthogonal, all terms in the sum are zero except those for which $i = j$. We thus have

$$\|v\|^2 = \sum_i \langle v_i, v_i \rangle = \sum_i \|v_i\|^2.$$

We can now take square roots to get the equation in the lemma. □

Lemma 12.8. *Any strictly orthogonal sequence is linearly independent.*

Proof. Let $\mathcal{V} = v_1,\dots,v_n$ be a strictly orthogonal sequence, and suppose we have a linear relation $\lambda_1 v_1 + \dots + \lambda_n v_n = 0$. For each i it follows that

$$\langle v_i, \lambda_1 v_1 + \dots + \lambda_n v_n \rangle = \langle v_i, 0 \rangle = 0.$$

The left hand side here is just

$$\lambda_1 \langle v_i, v_1 \rangle + \lambda_2 \langle v_i, v_2 \rangle + \dots + \lambda_n \langle v_i, v_n \rangle.$$

Moreover, the sequence $\mathcal{V}$ is orthogonal, so the inner products $\langle v_i, v_j \rangle$ are zero unless $j = i$, so the only nonzero term on the left hand side is $\lambda_i \langle v_i, v_i \rangle$, so we conclude that $\lambda_i \langle v_i, v_i \rangle = 0$. Moreover, the sequence is *strictly* orthogonal, so $v_i \neq 0$, so $\langle v_i, v_i \rangle > 0$. It follows that we must have $\lambda_i = 0$, so our original linear relation was the trivial one. We conclude that $\mathcal{V}$ is linearly independent, as claimed. □

Proposition 12.9. *Let V be a vector space with inner product, and let W be a subspace. Suppose that we have a strictly orthogonal sequence $\mathcal{W} = w_1,\dots,w_p$ that spans W, and we define*

$$\pi(v) = \frac{\langle v, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 + \dots + \frac{\langle v, w_p \rangle}{\langle w_p, w_p \rangle} w_p$$

(for all $v \in V$). Then $\pi(v) \in W$ and $v - \pi(v) \in W^\perp$, so $v = \pi(v) + (v - \pi(v)) \in W + W^\perp$. In particular, we have $W + W^\perp = V$, so W is complemented.

Remark 12.10. If the sequence $\mathcal{W}$ is orthonormal, then of course we have $\langle w_k, w_k \rangle = 1$ and the formula reduces to

$$\pi(v) = \langle v, w_1 \rangle w_1 + \ldots + \langle v, w_p \rangle w_p.$$

Proof. First note that the coefficients $\lambda_i = \langle v, w_i \rangle / \langle w_i, w_i \rangle$ are just numbers, so the element $\pi(v) = \lambda_1 w_1 + \ldots + \lambda_p w_p$ lies in the span of $w_1,\dots,w_p$, which is W. Next, we have

$$\langle w_i, \pi(v) \rangle = \lambda_1 \langle w_i, w_1 \rangle + \ldots + \lambda_i \langle w_i, w_i \rangle + \ldots + \lambda_p \langle w_i, w_p \rangle.$$

As the sequence $\mathcal{W}$ is orthogonal, we have $\langle w_i, w_j \rangle = 0$ for $j \neq i$, so only the i'th term in the above sum is nonzero. This means that

$$\langle w_i, \pi(v) \rangle = \lambda_i \langle w_i, w_i \rangle = \frac{\langle v, w_i \rangle}{\langle w_i, w_i \rangle} \langle w_i, w_i \rangle = \langle v, w_i \rangle = \langle w_i, v \rangle,$$

so $\langle w_i, v - \pi(v) \rangle = \langle w_i, v \rangle - \langle w_i, \pi(v) \rangle = 0$. As this holds for all i, and the elements w_i span W, we see that $\langle w, v - \pi(v) \rangle = 0$ for all $w \in W$, or in other words, that $v - \pi(v) \in W^\perp$, as claimed. □

Corollary 12.11 (Parseval's inequality)**.** *Let V be a vector space with inner product, and let $\mathcal{W} = w_1,\dots,w_p$ be an orthonormal sequence in V. Then for any $v \in V$ we have*

$$\|v\|^2 \geq \sum_{i=1}^{p} \langle v, w_i \rangle^2.$$

Moreover, this inequality is actually an equality iff $v \in \mathrm{span}(\mathcal{W})$.

Proof. Put $W = \mathrm{span}(\mathcal{W})$, and put $\pi(v) = \sum_{i=1}^{p} \langle v, w_i \rangle w_i$ as in Proposition 12.9. Put $\epsilon(v) = v - \pi(v)$, which lies in $W^\perp$. The sequence

$$\langle v, w_1 \rangle w_1, \dots, \langle v, w_p \rangle w_p, \epsilon(v)$$

is orthogonal, and the sum of the sequence is $\pi(v) + \epsilon(v) = v$. Lemma 12.7 therefore tells us that

$$\|v\|^2 = \|\langle v, w_1 \rangle w_1\|^2 + \dots + \|\langle v, w_p \rangle w_p\|^2 + \|\epsilon(v)\|^2 = \|\epsilon(v)\|^2 + \sum_i \langle v, w_i \rangle^2.$$

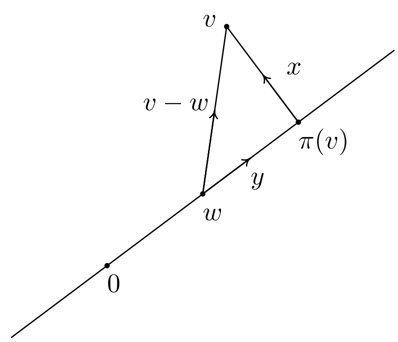
All terms here are nonnegative, so $\|v\|^2 \geq \sum_i \langle v, w_i \rangle^2$, with equality iff $\|\epsilon(v)\|^2 = 0$. Moreover, we have $\|\epsilon(v)\|^2 = 0$ iff $\epsilon(v) = 0$ iff $v = \pi(v)$ iff $v \in W$. □

Proposition 12.12. *Let $\mathcal{W}$ and π be as in Proposition 12.9. Then $\pi(v)$ is the point in W that is closest to v.*

Proof. Put $x = v - \pi(v)$, so $x \in W^\perp$. The distance from v to $\pi(v)$ is just $\|v - \pi(v)\| = \|x\|$. Now consider another point $w \in W$, with $w \neq \pi(v)$. The distance from v to w is just $\|v - w\|$; we must show that this is larger than $\|x\|$. Put $y = \pi(v) - w$, and note that $v - w = \pi(v) + x - w = x + y$. Note also that $y \in W$ (because $\pi(v) \in W$ and $w \in W$) and $x \in W^\perp$, so $\langle x, y\rangle = 0 = \langle y, x\rangle$. Finally, note that $y \neq 0$ and so $\|y\| > 0$. It follows that

$$\begin{aligned}\|v - w\|^2 = \|x + y\|^2 &= \langle x + y, x + y\rangle \\ &= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\ &= \|x\|^2 + 0 + 0 + \|y\|^2 > \|x\|^2.\end{aligned}$$

This shows that $\|v - w\| > \|x\| = \|v - \pi(v)\|$, so w is further from v than $\pi(v)$ is.



□

Theorem 12.13. *Let V be a vector space with inner product, and let $\mathcal{U} = u_1, \dots, u_n$ be a linearly independent list of elements of V. Then there is a strictly orthogonal sequence $\mathcal{V} = v_1, \dots, v_n$ such that* $\text{span}(v_1, \dots, v_i) = \text{span}(u_1, \dots, u_i)$ *for all i.*

Proof. The sequence $\mathcal{V}$ is generated by the *Gram-Schmidt procedure*, which we now describe. Put $U_i = \text{span}(u_1, \dots, u_i)$. We will construct the elements v_i by induction. For the initial step, we take $v_1 = u_1$, so (v_1) is an orthogonal basis for U_1. Suppose we have constructed an orthogonal basis $v_1, \dots, v_{i-1}$ for U_{i-1}. Proposition 12.9 then tells us that U_{i-1} is complemented, so $V = U_{i-1}^\perp + U_{i-1}$. In particular, we can write $u_i = v_i + w_i$ with $v_i \in U_{i-1}^\perp$ and $w_i \in U_{i-1}$. Explicitly, the formulae are

$$\begin{aligned} w_i &= \sum_{j=1}^{i-1} \frac{\langle u_i, v_j\rangle}{\langle v_j, v_j\rangle} v_j \\ v_i &= u_i - w_i. \end{aligned}$$

As $v_i \in U_{i-1}^\perp$ and $v_1, \dots, v_{i-1} \in U_{i-1}$, we have $\langle v_i, v_j\rangle = 0$ for $j < i$, so $(v_1, \dots, v_i)$ is an orthogonal sequence.

Next, note that $U_i = U_{i-1} + \mathbb{R}u_i$. As $u_i = v_i + w_i$ with $w_i \in U_{i-1}$, we see that this is the same as $U_{i-1} + \mathbb{R}v_i$. By our induction hypothesis, we have $U_{i-1} = \text{span}(v_1, \dots, v_{i-1})$, and it follows that $U_i = U_{i-1} + \mathbb{R}v_i = \text{span}(v_1, \dots, v_i)$. This means that $v_1, \dots, v_i$ is a spanning set of the i-dimensional space U_i, so it must be a basis. □

Corollary 12.14. *If V and $\mathcal{U}$ are as above, then there is an* orthonormal *sequence $\hat{v}_1, \dots, \hat{v}_n$ with* $\text{span}(\hat{v}_1, \dots, \hat{v}_i) = \text{span}(u_1, \dots, u_i)$ *for all i.*

Proof. Just find a strictly orthogonal sequence $v_1, \dots, v_n$ as in the Proposition, and put $\hat{v}_i = v_i / \|v_i\|$ as in Remark 12.4. □

Example 12.15. Consider the following elements of $\mathbb{R}^5$:

$$u_1 = \begin{bmatrix}1\\1\\0\\0\\0\end{bmatrix} \qquad u_2 = \begin{bmatrix}0\\1\\1\\0\\0\end{bmatrix} \qquad u_3 = \begin{bmatrix}0\\0\\1\\1\\0\end{bmatrix} \qquad u_4 = \begin{bmatrix}0\\0\\0\\1\\1\end{bmatrix}.$$

We apply the Gram-Schmidt procedure to get an orthogonal basis for the space $U = \text{span}(u_1, u_2, u_3, u_4)$. We have $v_1 = u_1 = \left[\begin{smallmatrix}1 & 1 & 0 & 0 & 0\end{smallmatrix}\right]^T$, so $\langle v_1, v_1\rangle = 2$ and $\langle u_2, v_1\rangle = 1$. Next, we have

$$v_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\langle v_1, v_1\rangle} v_1 = \begin{bmatrix}0\\1\\1\\0\\0\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\1\\0\\0\\0\end{bmatrix} = \begin{bmatrix}-1/2\\1/2\\1\\0\\0\end{bmatrix}.$$

It follows that $\langle v_2, v_2\rangle = 3/2$ and $\langle u_3, v_2\rangle = 1$, whereas $\langle u_3, v_1\rangle = 0$. It follows that

$$v_3 = u_3 - \frac{\langle u_3, v_1\rangle}{\langle v_1, v_1\rangle} v_1 - \frac{\langle u_3, v_2\rangle}{\langle v_2, v_2\rangle} v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{3/2} \begin{bmatrix} -1/2 \\ 1/2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -1/3 \\ 1/3 \\ 1 \\ 0 \end{bmatrix}.$$

It now follows that $\langle v_3, v_3\rangle = 4/3$ and $\langle u_4, v_3\rangle = 1$, whereas $\langle u_4, v_1\rangle = \langle u_4, v_2\rangle = 0$. It follows that

$$v_4 = u_4 - \frac{\langle u_4, v_1\rangle}{\langle v_1, v_1\rangle} v_1 - \frac{\langle u_4, v_2\rangle}{\langle v_2, v_2\rangle} v_2 - \frac{\langle u_4, v_3\rangle}{\langle v_3, v_3\rangle} v_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{4/3} \begin{bmatrix} 1/3 \\ -1/3 \\ 1/3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/4 \\ 1/4 \\ -1/4 \\ 1/4 \\ 1 \end{bmatrix}.$$

In conclusion, we have

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} -1/2 \\ 1/2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 1/3 \\ -1/3 \\ 1/3 \\ 1 \\ 0 \end{bmatrix} \qquad v_4 = \begin{bmatrix} -1/4 \\ 1/4 \\ -1/4 \\ 1/4 \\ 1 \end{bmatrix}.$$

Example 12.16. Consider the space $V = \mathbb{R}[x]_{\leq 2}$ with the inner product $\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)\,dx$. We will apply the Gram-Schmidt procedure to the usual basis $1, x, x^2$ to get an orthonormal basis for V. We start with $v_1 = u_1 = 1$, and note that $\langle v_1, v_1\rangle = \int_{-1}^{1} 1\,dx = 2$. We also have $\langle x, v_1\rangle = \int_{-1}^{1} x\,dx = \left[x^2/2\right]_{-1}^{1} = 0$, so x is already orthogonal to v_1. It follows that

$$v_2 = x - \frac{\langle x, v_1\rangle}{\langle v_1, v_1\rangle} v_1 = x,$$

and thus that $\langle v_2, v_2\rangle = \int_{-1}^{1} x^2\,dx = \left[x^3/3\right]_{-1}^{1} = 2/3$. We also have

$$\langle x^2, v_1\rangle = \int_{-1}^{1} x^2\,dx = 2/3$$
$$\langle x^2, v_2\rangle = \int_{-1}^{1} x^3\,dx = 0$$

so

$$v_3 = x^2 - \frac{\langle x^2, v_1\rangle}{\langle v_1, v_1\rangle} v_1 - \frac{\langle x^2, v_2\rangle}{\langle v_2, v_2\rangle} v_2 = x^2 - \frac{2/3}{2} 1 = x^2 - 1/3.$$

We find that

$$\langle v_3, v_3\rangle = \int_{-1}^{1} (x^2 - 1/3)^2\,dx = \int_{-1}^{1} x^4 - \tfrac{2}{3}x^2 + \tfrac{1}{9}\,dx = \left[\tfrac{1}{5}x^5 - \tfrac{2}{9}x^3 + \tfrac{1}{9}x\right]_{-1}^{1} = 8/45.$$

The required orthonormal basis is thus given by

$$\hat{v}_1 = v_1/\|v_1\| = 1/\sqrt{2}$$
$$\hat{v}_2 = v_2/\|v_2\| = \sqrt{3/2}x$$
$$\hat{v}_3 = v_3/\|v_3\| = \sqrt{45/8}(x^2 - 1/3).$$

Example 12.17. Consider the matrix $P = \left[\begin{smallmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]$, and let V be the space of 3×3 symmetric matrices of trace zero. We will find the matrix $Q \in V$ that is closest to P.

The general form of a matrix in V is

$$A = \left[\begin{smallmatrix} a & b & c \\ b & d & e \\ c & e & -a-d \end{smallmatrix}\right].$$

Thus, if we put

$$A_1 = \left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right] \qquad A_2 = \left[\begin{smallmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right] \qquad A_3 = \left[\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right] \qquad A_4 = \left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right] \qquad A_5 = \left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{smallmatrix}\right],$$

we see that an arbitrary element $A \in V$ can be written uniquely as $aA_1 + bA_2 + cA_3 + dA_4 + eA_5$, so $A_1, \dots, A_5$ is a basis for V. It is not too far from being an orthonormal basis: we have $\langle A_i, A_i\rangle = 2$ for all i, and when $i \neq j$ we have

$\langle A_i, A_j\rangle = 0$ except for the case $\langle A_1, A_4\rangle = 1$. Thus, the Gram-Schmidt procedure works out as follows:

$$\begin{aligned}
B_1 &= A_1 \\
B_2 &= A_2 - \frac{\langle A_2, B_1\rangle}{\langle B_1, B_1\rangle} B_1 = A_2 \\
B_3 &= A_3 - \frac{\langle A_3, B_1\rangle}{\langle B_1, B_1\rangle} B_1 - \frac{\langle A_3, B_2\rangle}{\langle B_2, B_2\rangle} B_2 = A_3 \\
B_4 &= A_4 - \frac{\langle A_4, B_1\rangle}{\langle B_1, B_1\rangle} B_1 - \frac{\langle A_4, B_2\rangle}{\langle B_2, B_2\rangle} B_2 - \frac{\langle A_4, B_3\rangle}{\langle B_3, B_3\rangle} B_3 = A_4 - \frac{1}{2} B_1 \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1/2 \end{bmatrix} \\
B_5 &= A_5 - \frac{\langle A_5, B_1\rangle}{\langle B_1, B_1\rangle} B_1 - \frac{\langle A_5, B_2\rangle}{\langle B_2, B_2\rangle} B_2 - \frac{\langle A_5, B_3\rangle}{\langle B_3, B_3\rangle} B_3 - \frac{\langle A_5, B_4\rangle}{\langle B_4, B_4\rangle} B_4 = A_5.
\end{aligned}$$

We have $\|B_4\| = \sqrt{3/2}$ and $\|B_i\| = \sqrt{2}$ for all other i. After noting that $(1/2)/\sqrt{3/2} = 1/\sqrt{6}$, it follows that the following matrices give an orthonormal basis for V:

$$\widehat{B}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad \widehat{B}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \widehat{B}_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \widehat{B}_4 = \frac{1}{\sqrt{6}}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad \widehat{B}_5 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

According to Proposition 12.12, the matrix Q is given by $Q = \sum_{i=1}^{5} \langle P, B_i\rangle \langle B_i, B_i\rangle^{-1} B_i$. The relevant inner products are $\langle P, B_1\rangle = \langle P, B_2\rangle = \langle P, B_3\rangle = 1$ and $\langle P, B_4\rangle = -1/2$ and $\langle P, B_5\rangle = 0$. We also have $\langle B_1, B_1\rangle = \langle B_2, B_2\rangle = \langle B_3, B_3\rangle = 2$ and $\langle B_4, B_4\rangle = 3/2$, so

$$Q = \frac{1}{2}(B_1 + B_2 + B_3) + \frac{-1}{2}\frac{2}{3} B_4 = \begin{bmatrix} 2/3 & 1/2 & 1/2 \\ 1/2 & -1/3 & 0 \\ 1/2 & 0 & -1/3 \end{bmatrix}$$

13. Hermitian forms

We now briefly discuss the analogue of inner products for complex vector spaces. Given a complex number $z = x+iy$, we write $\overline{z}$ for the complex conjugate, which is $x - iy$.

Definition 13.1. Let V be a vector space over $\mathbb{C}$. A *Hermitian form* on V is a rule that gives a number $\langle u, v\rangle \in \mathbb{C}$ for each $u, v \in V$, with the following properties:

(a) $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$ for all $u, v, w \in V$.
(b) $\langle tu, v\rangle = t\langle u, v\rangle$ for all $u, v \in V$ and $t \in \mathbb{C}$.
(c) $\langle u, v\rangle = \overline{\langle v, u\rangle}$ for all $u, v \in V$. In particular, by taking $v = u$ we see that $\langle u, u\rangle = \overline{\langle u, u\rangle}$, so $\langle u, u\rangle$ is real.
(d) For all $u \in V$ we have $\langle u, u\rangle \geq 0$ (which is meaningful because $\langle u, u\rangle \in \mathbb{R}$), and $\langle u, u\rangle = 0$ iff $u = 0$.

Note that (b) and (c) together imply that $\langle u, tv\rangle = \overline{t}\langle u, v\rangle$.

Given an inner product, we will write $\|u\| = \sqrt{\langle u, u\rangle}$, and call this the *norm* of u. We say that u is a *unit vector* if $\|u\| = 1$.

Example 13.2. We can define a Hermitian form on $\mathbb{C}^n$ by

$$\langle \mathbf{u}, \mathbf{v}\rangle = u_1\overline{v_1} + \cdots + u_n\overline{v_n}.$$

This gives

$$\|\mathbf{u}\|^2 = \langle \mathbf{u}, \mathbf{u}\rangle = |u_1|^2 + \cdots + |u_n|^2.$$

Definition 13.3. For any $n \times m$ matrix A over $\mathbb{C}$, we let $A^\dagger$ be the complex conjugate of the transpose of A, so for example

$$\begin{bmatrix} 1+i & 2+i & 3+i \\ 4+i & 5+i & 6+i \end{bmatrix}^\dagger = \begin{bmatrix} 1-i & 4-i \\ 2-i & 5-i \\ 3-i & 6-i \end{bmatrix}.$$

The above Hermitian form on $\mathbb{C}^n$ can then be rewritten as

$$\langle \mathbf{u}, \mathbf{v}\rangle = \mathbf{v}^\dagger \mathbf{u} = \overline{\mathbf{u}^\dagger \mathbf{v}}.$$

Example 13.4. We can define a Hermitian form on $\mathbb{C}[t]$ by

$$\langle f, g\rangle = \int_0^1 f(t)\overline{g(t)}\, dt.$$

This gives

$$\|f\|^2 = \langle f, f\rangle = \int_0^1 |f(t)|^2\, dt.$$

Example 13.5. We can define a Hermitian form on $M_n\mathbb{C}$ by $\langle A, B\rangle = \operatorname{trace}(B^\dagger A)$. If we identify $M_n\mathbb{C}$ with $\mathbb{C}^{n^2}$ in the usual way, then this is just the same as the Hermitian form in Example 13.2.

Our earlier results about inner products are mostly also true for Hermitian forms, but they need to be adjusted slightly by putting complex conjugates or absolute value signs in appropriate places. We will not go through the proofs, but we will at least record some of the statements.

Theorem 13.6 (The Cauchy-Schwartz inequality)**.** *Let V be a vector space over $\mathbb{C}$ with a Hermitian form, and let v and w be elements of V. Then*

$$|\langle v, w\rangle| \leq \|v\|\,\|w\|,$$

with equality iff v and w are linearly dependent over $\mathbb{C}$. □

Lemma 13.7. *Let V be a vector space over $\mathbb{C}$ with a Hermitian form, let $v_1, \dots, v_n$ be an orthogonal sequence in V, and put $v = v_1 + \dots + v_n$. Then*

$$\|v\| = \sqrt{\|v_1\|^2 + \dots + \|v_n\|^2}. \quad \square$$

Proposition 13.8. *Let V be a vector space over $\mathbb{C}$ with a Hermitian form, and let $\mathcal{W} = w_1, \dots, w_p$ be an orthonormal sequence in V. Then for any $v \in V$ we have*

$$\|v\|^2 \geq \sum_{i=1}^{p} |\langle v, w_i\rangle|^2.$$

Moreover, this inequality is actually an equality iff $v \in \operatorname{span}(\mathcal{W})$. □

14. Adjoints of linear maps

Definition 14.1. Let V and W be real vector spaces with inner products (or complex vector spaces with Hermitian forms). Let $\phi\colon V \to W$ and $\psi\colon W \to V$ be linear maps (over $\mathbb{R}$ or $\mathbb{C}$ as appropriate). We say that ϕ is *adjoint* to ψ if we have $\langle \phi(v), w\rangle = \langle v, \psi(w)\rangle$ for all $v \in V$ and $w \in W$.

This is essentially a basis-free formulation of the operation of transposing a matrix, as we see from the following example.

Example 14.2. Let A be an $n \times m$ matrix over $\mathbb{R}$, giving a linear map $\phi_A\colon \mathbb{R}^m \to \mathbb{R}^n$ by $\phi_A(\mathbf{v}) = A\mathbf{v}$. The transpose of A is then an $m \times n$ matrix, giving a linear map $\phi_{A^T}\colon \mathbb{R}^n \to \mathbb{R}^m$. We claim that ϕ_{A^T} is adjoint to ϕ_A. This is easy to see using the formula $\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{x}^T\mathbf{y}$ as in Remark 10.4. Indeed, we have

$$\langle \phi_A(\mathbf{u}), \mathbf{v}\rangle = \langle A\mathbf{u}, \mathbf{v}\rangle = (A\mathbf{u})^T\mathbf{v} = \mathbf{u}^T A^T \mathbf{v} = \langle \mathbf{u}, A^T\mathbf{v}\rangle = \langle \mathbf{u}, \phi_{A^T}(\mathbf{v})\rangle,$$

as required.

Example 14.3. Let A be an $n \times m$ matrix over $\mathbb{C}$, giving a linear map $\phi_A\colon \mathbb{C}^m \to \mathbb{C}^n$ by $\phi_A(\mathbf{v}) = A\mathbf{v}$. Let $A^\dagger$ be the complex conjugate of A^T. Then $\phi_{A^\dagger}$ is adjoint to ϕ_A.

Example 14.4. Let V be the set of functions of the form $p(x)e^{-x^2/2}$, where $p(x)$ is a polynomial. We use the inner product

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)g(x)\,dx,$$

as in Example 10.10. If we have a function $f(x) = p(x)e^{-x^2/2}$ in V, we note that

$$f'(x) = p'(x)e^{-x^2} + p(x).(-2x).e^{-x^2} = (p'(x) - 2x\,p(x))e^{-x^2/2},$$

and $p'(x) - 2x\,p(x)$ is again a polynomial, so $f'(x) \in V$. We can thus define a linear map $D\colon V \to V$ by $D(f) = f'$. We claim that D is adjoint to $-D$. This is equivalent to the statement that for all f and g in V, we have $\langle D(f), g\rangle + \langle f, D(g)\rangle = 0$. This is true because

$$\begin{aligned}
\langle f', g\rangle + \langle f, g'\rangle &= \int_{-\infty}^{\infty} f'(x)g(x) + f(x)g'(x)\,dx \\
&= \int_{-\infty}^{\infty} \frac{d}{dx}(f(x)g(x))\,dx \\
&= [f(x)g(x)]_{-\infty}^{\infty} \\
&= \lim_{x\to+\infty} f(x)g(x) - \lim_{x\to-\infty} f(x)g(x).
\end{aligned}$$

Both limits here are zero, because the very rapid decrease of e^{-x^2} wipes out the much slower increase of the polynomial terms.

Example 14.5. Consider the vector spaces $\mathbb{R}[x]_{\leq 2}$ (with inner product $\langle f,g\rangle = \int_0^1 f(x)g(x)\,dx$) and $\mathbb{R}^2$ (with the usual inner product). Define maps $\phi\colon \mathbb{R}[x]_{\leq 2} \to \mathbb{R}^2$ and $\psi\colon \mathbb{R}^2 \to \mathbb{R}[x]_{\leq 2}$ by

$$\phi(f) = \begin{bmatrix} f(0) \\ f(1) \end{bmatrix}$$
$$\psi\begin{bmatrix} p \\ q \end{bmatrix} = (30p+30q)x^2 - (36p+24q)x + (9p+3q).$$

We claim that ϕ is adjoint to ψ. To check this, consider a quadratic polynomial $f(x) = ax^2+bx+c \in \mathbb{R}[x]_{\leq 2}$ and a vector $\mathbf{v} = \begin{bmatrix} p \\ q \end{bmatrix} \in \mathbb{R}^2$. Note that $f(0) = c$ and $f(1) = a+b+c$, so $\phi(f) = \begin{bmatrix} c \\ a+b+c \end{bmatrix}$. We must show that $\langle f, \psi(\mathbf{v})\rangle = \langle \phi(f), \mathbf{v}\rangle$, or in other words that

$$\int_0^1 (ax^2+bx+c)((30p+30q)x^2 - (36p+24q)x + (9p+3q))\,dx = p\,f(0) + q\,f(1) = pc + q(a+b+c).$$

This is a straightforward calculation, which can be done by hand or using Maple: entering

```
expand(
 int( (a*x^2+b*x+c)*
      ((30*p+30*q)*x^2 - (36*p+24*q)*x + (9*p+3*q)),
      x=0..1
 )
);
```

gives $cp + aq + bq + cq$, as required.

Proposition 14.6. *Let V and W be finite-dimensional real vector spaces with inner products (or complex vector spaces with Hermitian forms). Let $\phi\colon V \to W$ be a linear maps (over $\mathbb{R}$ or $\mathbb{C}$ as appropriate). Then there is a unique map $\psi\colon W \to V$ that is adjoint to ϕ. (We write $\psi = \phi^*$ in the real case, or $\psi = \phi^\dagger$ in the complex case.)*

Proof. We will prove the complex case; the real case is similar but slightly easier.

We first show that there is at most one adjoint. Suppose that ψ and ψ' are both adjoint to ϕ, so

$$\langle v, \psi(w)\rangle = \langle \phi(v), w\rangle = \langle v, \psi'(w)\rangle$$

for all $v \in V$ and $w \in W$. This means that $\langle v, \psi(w) - \psi'(w)\rangle = 0$ for all v and w. In particular, we can take $v = \psi(w) - \psi'(w)$, and we find that

$$\|\psi(w) - \psi'(w)\|^2 = \langle \psi(w) - \psi'(w), \psi(w) - \psi'(w)\rangle = 0,$$

so $\psi(w) = \psi'(w)$ for all w, so $\psi = \psi'$.

To show that there exists an adjoint, choose an orthonormal basis $\mathcal{V} = v_1, \dots, v_n$ for V, and define a linear map $\psi\colon W \to V$ by

$$\psi(w) = \sum_{j=1}^n \langle w, \phi(v_j)\rangle v_j.$$

Recall that $\langle x, \lambda y\rangle = \overline{\lambda}\langle x,y\rangle$, and that $\overline{\langle x,y\rangle} = \langle y,x\rangle$. Using these rules we find that

$$\begin{aligned}
\langle v_i, \psi(w)\rangle &= \sum_j \langle v_i, \langle w, \phi(v_j)\rangle v_j\rangle \\
&= \sum_j \overline{\langle w, \phi(v_j)\rangle}\langle v_i, v_j\rangle \\
&= \sum_j \langle \phi(v_j), w\rangle \langle v_i, v_j\rangle \\
&= \langle \phi(v_i), w\rangle.
\end{aligned}$$

(For the last equality, recall that $\mathcal{V}$ is orthonormal, so $\langle v_i, v_j\rangle = 0$ except when $j = i$, so only the i'th term in the sum is nonzero. The i'th term simplifies to $\langle \phi(v_i), w\rangle$, because $\langle v_i, v_i\rangle = \|v_i\|^2 = 1$.)

More generally, any element $v \in V$ can be written as $\sum_i x_i v_i$ for some $x_1, \dots, x_n \in \mathbb{C}$, and then we have

$$\begin{aligned}
\langle v, \psi(w) \rangle &= \sum_i x_i \langle v_i, \psi(w) \rangle \\
&= \sum_i x_i \langle \phi(v_i), w \rangle \\
&= \langle \phi\left(\sum_i x_i v_i\right), w \rangle \\
&= \langle \phi(v), w \rangle.
\end{aligned}$$

This shows that ψ is adjoint to ϕ, as required. □

15. Fourier theory

You will already have studied Fourier series in the Advanced Calculus course. Here we revisit these ideas from a more abstract point of view, in terms of angles and distances in a Hermitian space of periodic functions.

Definition 15.1. We say that a function $f \colon \mathbb{R} \to \mathbb{C}$ is *periodic* if $f(t + 2\pi) = f(t)$ for all $t \in \mathbb{R}$. We let P be the set of all continuous periodic functions from $\mathbb{R}$ to $\mathbb{C}$, which is a vector space over $\mathbb{C}$. We define an inner product on P by

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(t)\overline{g(t)}\, dt.$$

Some important elements of P are the functions e_n, s_n and c_n defined as follows:

$$\begin{aligned}
e_n(t) &= \exp(int) && (\text{for } n \in \mathbb{Z}) \\
s_n(t) &= \sin(nt) && (\text{for } n > 0) \\
c_n(t) &= \cos(nt) && (\text{for } n \geq 0)\ .
\end{aligned}$$

De Moivre's theorem tells us that

$$\begin{aligned}
e_n &= c_n + i\, s_n \\
s_n &= (e_n - e_{-n})/(2i) \\
c_n &= (e_n + e_{-n})/2.
\end{aligned}$$

Definition 15.2. We put

$$T_n = \operatorname{span}(\{e_k \mid -n \leq k \leq n\}) \leq P,$$

and note that $T_n \leq T_{n+1}$ for all n. We also let T denote the span of all the e_k's, or equivalently, the union of all the sets T_n. The elements of T are the functions $f \colon \mathbb{R} \to \mathbb{C}$ that can be written in the form

$$f(t) = \sum_{k=-n}^{n} a_k e_k(t) = \sum_{k=-n}^{n} a_k \exp(ikt)$$

for some $n > 0$ and some coefficients $a_{-n}, \dots, a_n \in \mathbb{C}$. Functions of this form are called *trigonometric polynomials* or *finite Fourier series*.

Proposition 15.3. *The sequence $e_{-n}, e_{-n+1}, \dots, e_{n-1}, e_n$ is an orthonormal basis for T_n (so $\dim(T_n) = 2n + 1$).*

Proof. For $m \neq k$ we have

$$\begin{aligned}
\langle e_k, e_m \rangle &= (2\pi)^{-1} \int_0^{2\pi} e_k(t)\overline{e_m(t)}\, dt = (2\pi)^{-1} \int_0^{2\pi} \exp(ikt)\exp(-imt)\, dt \\
&= (2\pi)^{-1} \int_0^{2\pi} \exp(i(k-m)t)\, dt = (2\pi)^{-1} \left[\frac{\exp(i(k-m)t)}{i(k-m)}\right]_0^{2\pi} \\
&= \frac{1}{2(k-m)\pi i}\left(e^{2(k-m)\pi i} - 1\right).
\end{aligned}$$

As $k - m$ is an integer, we have $e^{2(k-m)\pi i} = 1$ and so $\langle e_k, e_m \rangle = 0$. This shows that the sequence of e_k's is orthogonal. We also have

$$\begin{aligned}
\langle e_k, e_k \rangle &= (2\pi)^{-1} \int_0^{2\pi} e_k(t)\overline{e_k(t)}\, dt = (2\pi)^{-1} \int_0^{2\pi} \exp(2k\pi it)\exp(-2k\pi it)\, dt \\
&= (2\pi)^{-1} \int_0^{2\pi} 1\, dt = 1.
\end{aligned}$$

Our sequence is therefore orthonormal, and so linearly independent. It also spans T_n (by the definition of T_n), so it is a basis. □

Definition 15.4. For any $f \in P$, let $\pi_n(f)$ be the orthogonal projection of f in T_n, so

$$\pi_n(f) = \sum_{m=-n}^{n} \langle f, e_m \rangle e_m.$$

We also put $\epsilon_n(f) = f - \pi_n(f)$, so $f = \pi_n(f) + \epsilon_n(f)$, with $\pi_n(f) \in T_n$ and $\epsilon_n(f) \in T_n^{\perp}$ (by Proposition 12.9).

Proposition 15.5. *The sequence $\mathcal{C}_n = c_0, c_1, \ldots, c_n, s_1, \ldots, s_n$ is another orthogonal basis for T_n. It is not orthonormal, but instead satisfies $\|s_k\|^2 = 1/2 = \|c_k\|^2$ for $k > 0$, and $\|c_0\|^2 = 1$.*

Proof. We use the identities

$$\begin{aligned} s_m &= (e_m - e_{-m})/(2i) \\ c_m &= (e_m + e_{-m})/2. \end{aligned}$$

If $k \neq m$ (with $k, m \geq 0$) we see that e_k and e_{-k} are orthogonal to e_m and e_{-m}. It follows that

$$\langle s_m, s_k \rangle = \langle s_m, c_k \rangle = \langle c_m, s_k \rangle = \langle c_m, c_k \rangle = 0.$$

Now suppose that $m > 0$, so c_m and s_m are both in the claimed basis. We have $\langle e_m, e_{-m} \rangle = 0$, and so

$$\langle s_m, c_m \rangle = \tfrac{1}{4i}\langle e_m - e_{-m}, e_m + e_{-m} \rangle = \tfrac{1}{4i}(\langle e_m, e_m \rangle + \langle e_m, e_{-m} \rangle + -\langle e_{-m}, e_m \rangle - \langle e_{-m}, e_{-m} \rangle) = \tfrac{1}{4i}(1 + 0 - 0 - 1) = 0.$$

This shows that $\mathcal{C}_n$ is an orthogonal sequence. For $k > 0$ we have

$$\begin{aligned} \langle s_k, s_k \rangle &= \tfrac{1}{2i}\tfrac{1}{2i}\langle e_k - e_{-k}, e_k - e_{-k} \rangle \\ &= \tfrac{1}{4}(1 - 0 - 0 + 1) = 1/2. \end{aligned}$$

Similarly, we have $\langle c_k, c_k \rangle = 1/2$. In the special case $k = 0$ we instead have $c_0(t) = 1$ for all t, so $\langle c_0, c_0 \rangle = (2\pi)^{-1} \int_0^{2\pi} 1\, dt = 1$. □

Corollary 15.6. *Using Proposition 12.9, we deduce that*

$$\pi_n(f) = \langle f, c_0 \rangle c_0 + 2\sum_{k=1}^{n} \langle f, c_k \rangle c_k + 2\sum_{k=1}^{n} \langle f, s_k \rangle s_k.$$

Theorem 15.7. *For any $f \in P$ we have $\|\epsilon_n(f)\| \to 0$ as $n \to \infty$.*

Proof. See Appendix A in the online version of the notes. (The proof is not examinable and will not be covered in lectures.) □

Remark 15.8. Recall that $\pi_n(f)$ is the closes point to f lying in T_n, so the number $\|\epsilon_n(f)\| = \|f - \pi_n(f)\|$ can be regarded as the distance from f to T_n. The theorem says that by taking n to be sufficiently large, we can make this distance as small as we like. In other words, f can be very well approximated by a trigonometric polynomial of sufficiently high degree.

Corollary 15.9. *For any $f \in P$ we have*

$$\|f\|^2 = \sum_{k=-\infty}^{\infty} |\langle f, e_k \rangle|^2 = |\langle f, c_0 \rangle|^2 + 2\sum_{k=1}^{\infty} |\langle f, c_k \rangle|^2 + 2\sum_{k=1}^{\infty} |\langle f, s_k \rangle|^2$$

Proof. As $e_{-n}, \ldots, e_n$ is an orthonormal basis for T_n, we have

$$\|f\|^2 - \|\epsilon_n(f)\|^2 = \|\pi_n(f)\|^2 = \|\sum_{k=-n}^{n} \langle f, e_k \rangle e_k\|^2 = \sum_{k=-n}^{n} |\langle f, e_k \rangle|^2$$

By taking limits as n tends to infinity, we see that $\|f\|^2 = \sum_{k=-\infty}^{\infty} |\langle f, e_k \rangle|^2$. Similarly, using Corollary 15.6 and Proposition 15.5, we see that

$$\begin{aligned} \|\pi_n(f)\|^2 &= |\langle f, c_0 \rangle|^2 \|c_0\|^2 + \sum_{k=1}^{n} 4|\langle f, c_k \rangle|^2 \|c_k\|^2 + \sum_{k=1}^{n} 4|\langle f, s_k \rangle|^2 \|s_k\|^2 \\ &= |\langle f, c_0 \rangle|^2 + 2\sum_{k=1}^{n} |\langle f, c_k \rangle|^2 + 2\sum_{k=1}^{n} |\langle f, s_k \rangle|^2 \end{aligned}$$

We can again let n tend to infinity to see that

$$\|f\|^2 = |\langle f, c_0\rangle|^2 + 2\sum_{k=1}^{\infty} |\langle f, c_k\rangle|^2 + 2\sum_{k=1}^{\infty} |\langle f, s_k\rangle|^2.$$

□

16. Diagonalisation of self-adjoint operators

Definition 16.1. Let V be a finite-dimensional vector space over $\mathbb{C}$. A *self-adjoint operator* on V is a linear map $\alpha\colon V \to V$ such that $\alpha^\dagger = \alpha$.

Theorem 16.2. *If $\alpha\colon V \to V$ is a self-adjoint operator, then every eigenvalue of α is real.*

Proof. First suppose that λ is an eigenvalue of α, so there exists a nonzero vector $v \in V$ with $\alpha(v) = \lambda v$. We then have

$$\lambda\langle v, v\rangle = \langle \lambda v, v\rangle = \langle \alpha(v), v\rangle = \langle v, \alpha^\dagger(v)\rangle = \langle v, \alpha(v)\rangle = \langle v, \lambda v\rangle = \overline{\lambda}\langle v, v\rangle.$$

As $v \neq 0$ we have $\langle v, v\rangle > 0$, so we can divide by this to see that $\lambda = \overline{\lambda}$, which means that λ is real. □

Theorem 16.3. *If $\alpha\colon V \to V$ is a self-adjoint operator, then one can choose an orthonormal basis $\mathcal{V} = v_1, \dots, v_n$ for V such that each v_i is an eigenvector of α.*

The following lemma will be useful in the proof.

Lemma 16.4. *Let $\alpha\colon V \to V$ be a self-adjoint operator, and let $W \leq V$ be a subspace such that $\alpha(W) \leq W$ (ie $\alpha(w) \in W$ for all $w \in W$). Then $\alpha(W^\perp) \leq W^\perp$.*

Proof. Suppose that $v \in W^\perp$; we must show that $\alpha(v)$ is also in $W^\perp$. To see this, consider $w \in W$, and note that $\langle \alpha(v), w\rangle = \langle v, \alpha^\dagger(w)\rangle = \langle v, \alpha(w)\rangle$ (by the definition of adjoints and the fact that $\alpha^\dagger = \alpha$). As $\alpha(W) \leq W$ we see that $\alpha(w) \in W$, so $\langle v, \alpha(w)\rangle = 0$ (because $v \in W^\perp$). We conclude that $\langle \alpha(v), w\rangle = 0$ for all $w \in W$, so $\alpha(v) \in W^\perp$ as claimed. □

Proof of Theorem 16.3. Put $n = \dim(V)$; the proof is by induction on n. If $n = 1$ then we choose any unit vector $v_1 \in V$ and note that $V = \mathbb{C}v_1$. This means that $\alpha(v_1) = \lambda_1 v_1$ for some $\lambda_1 \in \mathbb{C}$, so v_1 is an eigenvector, and this proves the theorem in the case $n = 1$.

Now suppose that $n > 1$. The characteristic polynomial of α is then a polynomial of degree n over $\mathbb{C}$, so it must have at least one root (by the fundamental theorem of algebra), say λ_1. We know that the roots of the characteristic polynomial are precisely the eigenvalues, so λ_1 is an eigenvalue, so we can find a nonzero vector $u_1 \in V$ with $\alpha(u_1) = \lambda_1 u_1$. We then put $v_1 = u_1/\|u_1\|$, so $\|v_1\| = 1$ and v_1 is still an eigenvector of eigenvalue λ_1, which implies that $\alpha(\mathbb{C}v_1) \leq \mathbb{C}v_1$. Now put $V' = (\mathbb{C}v_1)^\perp$. The lemma tells us that $\alpha(V') \leq V'$, so we can regard α as a self-adjoint operator on V'. Moreover, $\dim(V') = n - 1$, so our induction hypothesis applies. This means that there is an orthonormal basis for V' (say $v_2, v_3, \dots, v_n$) consisting of eigenvectors for α. It follows that $v_1, v_2, \dots, v_n$ is an orthonormal basis for V consisting of eigenvectors for α. □

Appendix A. Féjer's Theorem

In this appendix, we will outline a proof of Theorem 15.7: for any $f \in P$, we have $\|\epsilon_n(f)\| \to 0$ as $n \to \infty$.

For $f \in P$ and $n > 0$, we put

$$\theta_n(f) = (\pi_0(f) + \cdots + \pi_{n-1}(f))/n.$$
$$\delta_n(f) = f - \theta_n(f).$$

Theorem A.1 (Féjer's Theorem)**.** *For any $f \in P$, we have*

$$\max\{|\delta_n(f)(x)| \mid x \in \mathbb{R}\} \to 0$$

as $n \to \infty$.

We will sketch the proof of this shortly. First, however, we explain why Theorem A.1 implies Theorem 15.7.

Suppose we have $g \in P$, and put $m = \max\{|g(x)| \mid x \in \mathbb{R}\}$, so for all x we have $0 \leq |g(x)|^2 \leq m^2$. Then

$$\|g\|^2 = \frac{1}{2\pi}\int_0^{2\pi} |g(x)|^2\, dx \leq \frac{1}{2\pi}\int_0^{2\pi} m^2\, dx = m^2,$$

so $\|g\| \leq m$. Taking $g = \delta_n(f)$, we see that

$$0 \leq \|\delta_n(f)\| \leq \max\{|\delta_n(f)(x)| \mid x \in \mathbb{R}\}.$$

Using Féjer's Theorem and the Sandwich Lemma, we deduce that $\|\delta_n(f)\| \to 0$ as $n \to \infty$.

We now need to relate δ_n to ϵ_n. Note that for $k \leq n$ we have $\pi_k(f) \in T_k \leq T_n$. It follows that $\theta_n(f) \in T_n$. We also know (from Proposition 12.12) that $\pi_n(f)$ is the closest point to f in T_n. In other words, for any $g \in T_n$, we have $\|f - \pi_n(f)\| \leq \|f - g\|$. In particular, we can take $g = \theta_n(f)$ to see that $\|f - \pi_n(f)\| \leq \|f - \theta_n(f)\|$, or in other words $\|\epsilon_n(f)\| \leq \|\delta_n(f)\|$. As $\|\delta_n(f)\| \to 0$, the Sandwich Lemma again tells us that $\|\epsilon_n(f)\| \to 0$, proving Theorem 15.7.

The proof of Féjer's Theorem depends on the properties of certain functions $d_n(t)$ and $k_n(t)$ (called the Dirichlet kernel and the Féjer kernel) which are defined as follows:

$$d_n(t) = \sum_{j=-n}^{n} e_j(t)$$

$$k_n(t) = (\sum_{m=0}^{n-1} d_m(t))/n.$$

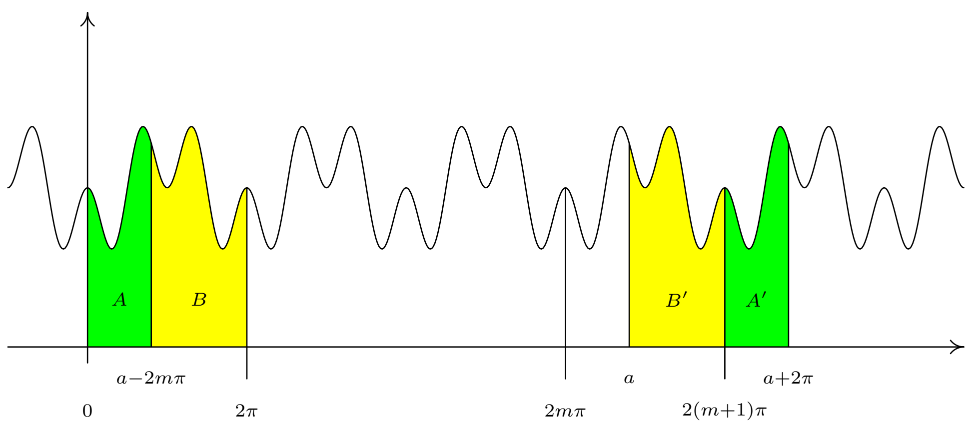
Proposition A.2. *If $g = \pi_n(f)$ and $h = \theta_n(f)$, then*

$$g(t) = (2\pi)^{-1} \int_{s=-\pi}^{\pi} f(t+s) d_n(-s)\, ds$$

$$h(t) = (2\pi)^{-1} \int_{s=-\pi}^{\pi} f(t+s) k_n(-s)\, ds.$$

Lemma A.3. *If u is a periodic function and $a \in \mathbb{R}$, then $\int_a^{a+2\pi} u(t)\, dt = \int_0^{2\pi} u(t)\, dt$.*

Proof. Let $2m\pi$ be the largest multiple of 2π that is less than or equal to a. Assuming that u is real and positive, we have a picture like this:



We have $\int_0^{2\pi} u(t)\, dt = \text{area}(A) + \text{area}(B)$ and $\int_a^{a+2\pi} u(t)\, dt = \text{area}(B') + \text{area}(A')$, but clearly $\text{area}(A) = \text{area}(A')$ and $\text{area}(B) = \text{area}(B')$, and the claim follows. Much the same argument works even if u is not real and positive, but one needs equations rather than pictures. □

Proof of Proposition A.2. Consider the integral $I = (2\pi)^{-1} \int_{-\pi}^{\pi} f(t+s) d_n(-s)\, ds$. We put $x = t + s$, so $s = x - t$ and $ds = dx$ and the endpoints $s = \pm\pi$ become $x = t \pm \pi$, so $I = (2\pi)^{-1} \int_{t-\pi}^{t+\pi} f(x) d_n(t-x)\, dx$. Next, we can use the lemma to convert this to $I = (2\pi)^{-1} \int_0^{2\pi} f(x) d_n(t-x)\, dx$. We then note that

$$e_k(t-x) = \exp(2\pi i k(t-x)) = \exp(2\pi i k t)\overline{\exp(2\pi i k x)} = e_k(t)\overline{e_k(x)},$$

so

$$d_n(t-x) = \sum_{k=-n}^{n} e_k(t-x) = \sum_{k=-n}^{n} e_k(t)\overline{e_k(x)},$$

so

$$\begin{aligned} I &= \sum_{k=-n}^{n} (2\pi)^{-1} \int_0^{2\pi} f(x)\overline{e_k(x)} e_k(t)\, dx \\ &= \sum_{k=-n}^{n} \langle f, e_k \rangle e_k(t) \\ &= \pi_n(f)(t) = g(t), \end{aligned}$$

as claimed. Next, we recall that $k_n(t) = (\sum_{m=0}^{n-1} d_m(t))/n$, so

$$(2\pi)^{-1}\int_{-\pi}^{\pi} f(t+s)k_n(-s)\,ds = \frac{1}{n}\sum_{j=0}^{n-1}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t+s)d_j(-s)\,ds$$
$$= \frac{1}{n}\sum_{j=0}^{n-1}\pi_j(f)(t) = \theta_n(f)(t) = h(t),$$

as required. □

Corollary A.4. $\int_{-\pi}^{\pi} k_n(s)\,ds = 2\pi$.

Proof. We take $f = e_0$ in Proposition A.2. We have $\pi_k(e_0) = e_0$ for all k, and so $h = \theta_n(e_0) = (e_0 + \cdots + e_0)/n = e_0$. Thus, the proposition tells us that

$$e_0(t) = (2\pi)^{-1}\int_{s=-\pi}^{\pi} e_0(t+s)k_n(-s)\,ds.$$

However, $e_0(x) = 1$ for all x, so this simplifies to

$$1 = (2\pi)^{-1}\int_{-\pi}^{\pi} k_n(-s)\,ds.$$

Moreover, we have $e_j(-s) = e_{-j}(s)$, and it follows from this that $d_j(-s) = d_j(s)$ and $k_n(-s) = k_n(s)$. It therefore follows that

$$\int_{-\pi}^{\pi} k_n(s)\,ds = 2\pi,$$

as claimed. □

Proposition A.5.

$$k_n(t) = \frac{1}{n}\left(\frac{\sin(nt/2)}{\sin(t/2)}\right)^2.$$

Proof. We will focus on the case $n = 6$; the general case is the same, but needs more complicated notation.

Put $z = e^{i\pi t}$, so $e_j(t) = z^{2j}$. Put

$$p_n = (1 + z + z^2 + \cdots + z^{n-1})(1 + z^{-1} + z^{-2} + \cdots + z^{1-n})$$

We can expand this out and write the terms in an $n \times n$ square array, which looks like this in the case $n = 6$:

1	z	z^2	z^3	z^4	z^5
z^{-1}	1	z	z^2	z^3	z^4
z^{-2}	z^{-1}	1	z	z^2	z^3
z^{-3}	z^{-2}	z^{-1}	1	z	z^2
z^{-4}	z^{-3}	z^{-2}	z^{-1}	1	z
z^{-5}	z^{-4}	z^{-3}	z^{-2}	z^{-1}	1

We have divided the square into L-shaped blocks. The sum of the terms in the third block (for example) is

$$z^{-2} + z^{-1} + 1 + z + z^2 = e_{-2}(t) + e_{-1}(t) + e_0(t) + e_1(t) + e_2(t) = d_3(t).$$

More generally, the sums of the terms in the six different L-shaped blocks are $d_0(t), d_1(t), \ldots, d_5(t)$. Adding these together, we see that

$$p_n(t) = d_0(t) + d_1(t) + \cdots + d_{n-1}(t) = n\,k_n(t).$$

Now put $w = e^{\pi i t}$, so $z = w^2$ and $\sin(t/2) = (w - w^{-1})/(2i)$ and $\sin(nt/2) = (w^n - w^{-n})/(2i)$. On the other hand, we have the geometric progression formula

$$1 + z + \cdots + z^{n-1} = \frac{z^n - 1}{z - 1} = \frac{w^{2n} - 1}{w^2 - 1} = \frac{w^n}{w}\,\frac{(w^n - w^{-n})/(2i)}{(w - w^{-1})/(2i)}$$
$$= w^{n-1}\frac{\sin(nt/2)}{\sin(t/2)}$$

Similarly, we have

$$1 + z^{-1} + \cdots + z^{1-n} = w^{1-n} \frac{\sin(nt/2)}{\sin(t/2)}.$$

If we multiply these two equations together, we get

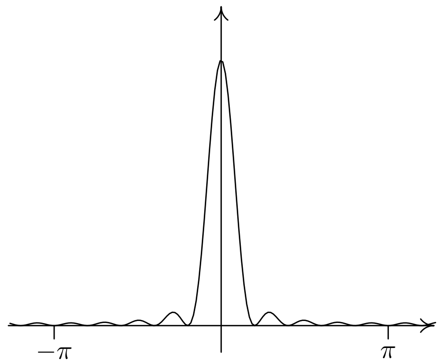
$$p_n(t) = \left(\frac{\sin(nt/2)}{\sin(t/2)} \right)^2.$$

Dividing by n gives

$$k_n(t) = \frac{1}{n} \left(\frac{\sin(nt/2)}{\sin(t/2)} \right)^2,$$

as claimed. □

It is now easy to plot $k_n(s)$. For $n = 10$, the picture is as follows:



There is a narrow spike (of width approximately $4\pi/n$) near $s = 0$, and $k_n(s)$ is small on the remainder of the interval $[-\pi, \pi]$. Now think what happens when we evaluate the integral

$$h(t) = (2\pi)^{-1} \int_{-\pi}^{\pi} f(t + s) k_n(-s)\, ds.$$

When s is very small, $f(t + s)$ is close to $f(t)$. If s is not very small, then $k_n(-s)$ is tiny and we do not get much of a contribution to the integral anyway. Thus, it will not make much difference if we replace $f(t + s)$ by $f(t)$. This gives

$$h(t) \approx (2\pi)^{-1} \int_{-\pi}^{\pi} f(t) k_n(-s)\, ds = f(t).(2\pi)^{-1} \int_{-\pi}^{\pi} k_n(-s)\, ds = f(t)$$

(where we have used Corollary A.4). All this can be made more precise to give an explicit upper bound for the quantity

$$\max\{\delta_n(f)(t) \mid t \in \mathbb{R}\} = \max\{|f(t) - h(t)| \mid t \in \mathbb{R}\},$$

which can be used to prove Theorem A.1.