Vector Spaces and Fourier Theory — Problem Sheet 1

The first two problems on this sheet relate to the week 1 lectures. The remaining questions are essentially revision of SOM201 (Linear Algebra for Applications). Please hand in questions 2 and 3 in the Monday lecture next week.

Exercise 1. For each of the following vector spaces V, write down two typical elements of V (say u and v), calculate $u+v$ and $10v$, and observe that these are again elements of V. (For example, in the case $V=\mathbb{R}^2$ we could take $u=\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]$ and $v=\left[\begin{smallmatrix}3\\4\end{smallmatrix}\right]$, then we would have $u+v=\left[\begin{smallmatrix}4\\6\end{smallmatrix}\right]$ and $10v=\left[\begin{smallmatrix}30\\40\end{smallmatrix}\right]$, both of which are again elements of $\mathbb{R}^2$.)

(a) $V=\mathbb{R}^4$ (b) $V=M_{2,3}(\mathbb{R})$ (c) $V=\mathbb{R}[x]$

(d) V is the set of physical vectors, as in Example 2.6 in the notes.

Exercise 2. Explain why none of the following is a vector space (with the obvious definition of addition and scalar multiplication).

(a) $V_0=\left\{\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right]\in M_2\mathbb{R}\mid a\leq b\leq c\leq d\right\}$
(b) $V_1=\left\{\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]\in\mathbb{R}^2\mid a+b \text{ is an odd integer }\right\}$
(c) $V_2=\left\{\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]\in\mathbb{R}^2\mid x^2=y^2\right\}$.
(d) $V_3=\{p\in\mathbb{R}[x]\mid p(0)p(1)=0\}$

Exercise 3. For each of the following lists of vectors, say (with justification) whether they are linearly independent, whether they span $\mathbb{R}^3$, and whether they form a basis of $\mathbb{R}^3$. (If you understand the concepts involved, you should be able to do this by eye, without any calculation.)

(a) $\mathbf{u}_1=\left[\begin{smallmatrix}1\\0\\2\end{smallmatrix}\right]$, $\mathbf{u}_2=\left[\begin{smallmatrix}3\\0\\4\end{smallmatrix}\right]$, $\mathbf{u}_3=\left[\begin{smallmatrix}5\\0\\6\end{smallmatrix}\right]$, $\mathbf{u}_4=\left[\begin{smallmatrix}7\\0\\8\end{smallmatrix}\right]$.
(b) $\mathbf{v}_1=\left[\begin{smallmatrix}1\\1\\0\end{smallmatrix}\right]$, $\mathbf{v}_2=\left[\begin{smallmatrix}1\\0\\1\end{smallmatrix}\right]$, $\mathbf{v}_3=\left[\begin{smallmatrix}0\\1\\1\end{smallmatrix}\right]$, $\mathbf{v}_4=\left[\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right]$.
(c) $\mathbf{w}_1=\left[\begin{smallmatrix}1\\2\\3\end{smallmatrix}\right]$, $\mathbf{w}_2=\left[\begin{smallmatrix}4\\5\\6\end{smallmatrix}\right]$.
(d) $\mathbf{x}_1=\left[\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right]$, $\mathbf{x}_2=\left[\begin{smallmatrix}0\\2\\2\end{smallmatrix}\right]$, $\mathbf{x}_3=\left[\begin{smallmatrix}0\\0\\4\end{smallmatrix}\right]$.

Exercise 4. Consider the following subspaces of $\mathbb{R}^4$:

$$U=\{[w,x,y,z]^T\mid w-x+y-z=0\}$$
$$V=\{[w,x,y,z]^T\mid w+x+y=0=x+y+z\}$$
$$W=\{[u,u+v,u+2v,u+3v]^T\mid u,v\in\mathbb{R}\}.$$

Find $U\cap V$, $U\cap W$ and $V\cap W$.

Exercise 5. Consider the planes P, Q and R in $\mathbb{R}^3$ given by

$$P=\{[x,y,z]^T\mid x+2y+3z=0\}$$
$$Q=\{[x,y,z]^T\mid 3x+2y+z=0\}$$
$$R=\{[x,y,z]^T\mid x+y+z=0\}.$$

This system of planes has an unusual feature, not shared by most other systems of three planes through the origin. What is it?

Exercise 6. Let a, b and c be nonzero real numbers. Show that the matrix

$$A=\begin{bmatrix}0&a&b\\-a&0&c\\-b&-c&0\end{bmatrix}$$

has one real eigenvalue, and two purely imaginary eigenvalues.

Exercise 7. Find the rank of the following matrix:

$$A=\begin{bmatrix}1&1&1&1\\1&1&-1&-1\\1&-1&0&0\\0&0&1&-1\end{bmatrix}$$