Vector Spaces and Fourier Theory — Problem Sheet 2

Please log on at `http://aim.shef.ac.uk/AiM` and enter your answers there. Use your usual user name (eg `pma04xyz`) but **not** your usual password. Instead, click the 'Password reminder' button and your password will be emailed to you.

Exercise 1. Which of the following rules defines a linear map?

(a) $\phi_0\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $\phi_0 \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} x+y \\ x-y \end{smallmatrix}\right]$
(b) $\phi_1\colon \mathbb{R}^3 \to \mathbb{R}$ given by $\phi_1 \left[\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right] = xyz$
(c) $\phi_2\colon M_2\mathbb{R} \to \mathbb{R}$ given by $\phi_2 \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = \max(|a|, |b|, |c|, |d|)$
(d) $\phi_3\colon \mathbb{R}[x] \to \mathbb{R}$ given by $\phi_3(f) = f(0) + f'(1) + f''(2)$
(e) $\phi_4\colon \mathbb{R}[x] \to \mathbb{R}$ given by $\phi_4(f) = f(0)f(1)$.

Exercise 2. In each of the cases below, give an example of a nonzero linear map $\phi\colon V \to W$. (Here "nonzero" means that there is at least one $v \in V$ such that $\phi(v) \neq 0$.)

(a) $V = \mathbb{R}^4$ and $W = \mathbb{R}^2$
(b) $V = M_3\mathbb{R}$ and $W = \mathbb{R}^2$
(c) $V = M_3\mathbb{R}$ and $W = \mathbb{R}[x]$
(d) $V = \mathbb{R}[x]$ and $W = M_2\mathbb{R}$

Exercise 3. Define $\chi\colon M_n\mathbb{R} \to \mathbb{R}[t]_{\leq n}$ by

$$\chi(A) = \det(tI - A) = \text{ the characteristic polynomial of } A.$$

Is this a linear map?

Exercise 4. Given a matrix $A \in M_n\mathbb{C}$, we write $\rho(A)$ for the *spectral radius* of A, which is the largest absolute value of any eigenvalue of A. In symbols, we have

$$\rho(A) = \max\{|\lambda| \mid \det(\lambda I - A) = 0\}.$$

Is $\rho\colon M_n\mathbb{C} \to \mathbb{C}$ a linear map?

Exercise 5. Which of the following subsets of $\mathbb{R}^4$ is a subspace?

$$U_0 = \{[w, x, y, z]^T \mid w + x = 0\}$$
$$U_1 = \{[w, x, y, z]^T \mid w + x = 1\}$$
$$U_2 = \{[w, x, y, z]^T \mid w + 2x + 3y + 4z = 0\}$$
$$U_3 = \{[w, x, y, z]^T \mid w + x^2 + y^3 + z^4 = 0\}$$
$$U_4 = \{[w, x, y, z]^T \mid w^2 + x^2 = 0\}$$

Exercise 6. Which of the following subsets of $F(\mathbb{R})$ are subspaces?

$$U_0 = \{f \mid f(0) = 0\} \qquad U_1 = \{f \mid f(1) = 1\}$$
$$U_2 = \{f \mid f(0) \geq 0\} \qquad U_3 = \{f \mid f(0) = f(1)\}$$
$$U_4 = \{f \mid f(0)f(1) = f(2)f(3)\}.$$

Exercise 7. For each of the following vector spaces V, give an example of a subspace $W \leq V$ such that $W \neq 0$ and $W \neq V$.

(a) $V = \mathbb{R}[x]_{\leq 3}$
(b) $V = M_{2,3}\mathbb{R}$
(c) $V = \{[x, y, z]^T \in \mathbb{R}^3 \mid x + y + z = 0\}$

Exercise 8. For each of the following vector spaces U, give an example of subspaces $V, W \leq U$ such that $V \neq 0$ and $W \neq 0$ but $V \cap W = 0$.

(a) $U = \mathbb{R}^4$
(b) $U = M_2\mathbb{R}$
(c) $U = \{[x, y, z]^T \in \mathbb{R}^3 \mid x + y + z = 0\}$