Vector Spaces and Fourier Theory — Problem Sheet 3

Please hand in questions 1 and 5 in the lecture on Monday 3rd March.

Exercise 1. Put $U = \mathbb{R}[x]_{\leq 2}$ and $V = \{f \in U \mid f(0) = 0\}$ and $W = \{f \in U \mid f(1) + f(-1) = 0\}$. Show that $V \cap W$ is the set of all polynomials of the form $f(x) = bx$, and that $V + W = U$. Please write your argument carefully, using complete sentences and correct notation.

Exercise 2. Define $\alpha\colon \mathbb{R}^2 \to M_2\mathbb{R}$ by $\alpha\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] = \left[\begin{smallmatrix} u & -u \\ -v & v \end{smallmatrix}\right]$. Show that α is injective, and that

$$\text{image}(\alpha) = \{A \in M_2\mathbb{R} \mid A\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] = 0\}.$$

Exercise 3. Define $\phi\colon M_2\mathbb{R} \to M_2\mathbb{R}$ by $\phi(A) = A - \frac{1}{2}\,\text{trace}(A)I$.

(a) Find $\phi\left(\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]\right)$.
(b) Show that $\ker(\phi) = \{aI \mid a \in \mathbb{R}\}$.
(c) Show that $\text{image}(\phi) = \{A \in M_2\mathbb{R} \mid \text{trace}(A) = 0\}$.

Exercise 4. Define $\phi\colon \mathbb{R}[x]_{\leq 2} \to \mathbb{R}^3$ by

$$\phi(f) = \left[\int_{-1}^{0} f(x)\,dx, \int_{-1}^{1} f(x)\,dx, \int_{0}^{1} f(x)\,dx\right]^T.$$

(a) If $f(x) = ax^2 + bx + c$, find $\phi(f)$.
(b) Show that $\ker(\phi) = \{c(1 - 3x^2) \mid c \in \mathbb{R}\}$.
(c) Find a function $g_+(x) = px + q$ such that $\phi(g_+) = [1, 1, 0]^T$.
(d) Put $g_-(x) = g_+(-x)$, and show that $\phi(g_-) = [0, 1, 1]^T$.
(e) Deduce that $\text{image}(\phi) = \{[u, v, w]^T \in \mathbb{R}^3 \mid v = u + w\}$.

Exercise 5. For each of the following linear maps, decide whether the map is injective, whether it is surjective, and whether it is an isomorphism. Please write your arguments carefully, using complete sentences and correct notation. Where counterexamples are required, make them as simple and specific as possible.

(a) $\phi\colon \mathbb{R}^2 \to \mathbb{R}^3$ given by $\phi\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} x \\ y \\ x \end{smallmatrix}\right]$
(b) $\phi\colon \mathbb{R}^3 \to \mathbb{R}^2$ given by $\phi\left[\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right] = \left[\begin{smallmatrix} x-y \\ y-z \end{smallmatrix}\right]$
(c) $\phi\colon \mathbb{R}[x]_{\leq 2} \to \mathbb{R}^3$ given by $\phi(f) = [f(0), f'(0), f''(0)]^T$
(d) $\phi\colon \mathbb{R}^2 \to M_2\mathbb{R}$ given by $\phi\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & x+y \\ x+y & 0 \end{smallmatrix}\right]$
(e) $\phi\colon \mathbb{R}[x] \to \mathbb{R}$ given by $\phi(f) = \int_{-1}^{1} f(x)\,dx$

Exercise 6. Put

$$L = \{\left[\begin{smallmatrix} s \\ 2s \end{smallmatrix}\right] \mid s \in \mathbb{R}\} \qquad M = \{\left[\begin{smallmatrix} 2t \\ t \end{smallmatrix}\right] \mid t \in \mathbb{R}\}$$

Show that $L \cap M = 0$ and $L + M = \mathbb{R}^2$ (or in other words, $\mathbb{R}^2 = L \oplus M$).

Exercise 7. Put $U = M_3\mathbb{R}$ and $V = \{A \in U \mid A^T = A\}$ and $W = \{A \in U \mid A^T = -A\}$. Show that $V \cap W = 0$ and $V + W = U$ (or in other words, $U = V \oplus W$).