Vector Spaces and Fourier Theory — Problem Sheet 4

There is an online test this week, which covers most of the questions on this sheet.

Exercise 1. Which of the following lists of vectors are linearly independent?

(a) $\mathbf{u}_1 = [1,0,0,0,1]^T, \mathbf{u}_2 = [0,2,0,2,0]^T, \mathbf{u}_3 = [0,0,3,0,0]^T$

(b) $\mathbf{v}_1 = [1,1,1,1]^T, \mathbf{v}_2 = [2,0,0,2]^T, \mathbf{v}_3 = [0,4,4,0]^T$

(c) $\mathbf{w}_1 = [1,1,2]^T, \mathbf{w}_2 = [4,5,7]^T, \mathbf{w}_3 = [1,1,1]^T$

Exercise 2. Consider the list $\mathcal{V} = 1, x, (1+x)^2, 1+x^2$ of elements of $\mathbb{R}[x]_{\leq 2}$.

(a) Simplify $\mu_{\mathcal{V}}([0,1,1,-1]^T)$.

(b) Find $\boldsymbol{\lambda} \in \mathbb{R}^4$ such that $\mu_{\mathcal{V}}(\boldsymbol{\lambda}) = x^2$.

(c) Find $\boldsymbol{\lambda} \in \mathbb{R}^4$ such that $\boldsymbol{\lambda} \neq 0$ but $\mu_{\mathcal{V}}(\boldsymbol{\lambda}) = 0$ (showing that $\mathcal{V}$ is linearly dependent).

Exercise 3. Which of the following lists of matrices spans $M_2\mathbb{R}$?

(a) $\mathcal{A} = \left[\begin{smallmatrix} 0 & 1 \\ 2 & 3 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 2 \\ 1 & 3 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 1 \\ 3 & 2 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 3 \\ 2 & 1 \end{smallmatrix}\right]$

(b) $\mathcal{B} = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$

(c) $\mathcal{C} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$

(d) $\mathcal{D} = \left[\begin{smallmatrix} 463 & 859 \\ 265 & -463 \end{smallmatrix}\right], \left[\begin{smallmatrix} 937 & 724 \\ 195 & -937 \end{smallmatrix}\right], \left[\begin{smallmatrix} 431 & 736 \\ 428 & -431 \end{smallmatrix}\right], \left[\begin{smallmatrix} 777 & 152 \\ 522 & -777 \end{smallmatrix}\right]$

Exercise 4. Put $r_k(x) = (x+k)^2$. Prove that the list $\mathcal{R} = r_0, r_1, r_2$ spans $\mathbb{R}[x]_{\leq 2}$.

Exercise 5. Suppose we have real numbers $a, b, c \in \mathbb{R}$ and functions $f, g, h \in C(\mathbb{R})$ such that

$$\begin{aligned} f(a) &= 1 & g(a) &= 0 & h(a) &= 0 \\ f(b) &= 0 & g(b) &= 1 & h(b) &= 0 \\ f(c) &= 0 & g(c) &= 0 & h(c) &= 1 \end{aligned}$$

Prove that f, g and h are linearly independent.

Exercise 6. Put $f_k(x) = e^{kx}$. Calculate $W(f_1, f_2, f_3)$. Are f_1, f_2 and f_3 linearly independent?

Exercise 7. Find and simplify the Wronskian of the functions $g_0(x) = x^n$, $g_1(x) = x^{n+1}$ and $g_2(x) = x^{n+2}$. You may want to use Maple for this. If you do, you will need to use `simplify(...)` to get the answer in its simplest form.

Exercise 8. Define a linear map $\phi\colon M_3\mathbb{R} \to \mathbb{R}[x]_{\leq 4}$ by

$$\phi(A) = [1, x, x^2]A[1, x, x^2]^T.$$

Show that ϕ is surjective, and find a basis for its kernel.

Exercise 9. (a) Define a map $\phi\colon \mathbb{R}[x]_{\leq 3} \to \mathbb{R}^2$ by $\phi(f) = [f(0), f(1)]^T$. Show that this is surjective, and that the kernel is spanned by $x^2 - x$ and $x^3 - x^2$.

(b) Define a map $\psi\colon \mathbb{R}[x]_{\leq 2} \to \mathbb{R}^4$ by $\psi(f) = [f(0), f(1), f(2), f(3)]^T$. Show that this is injective, and that the image is the space

$$V = \{[u_0, u_1, u_2, u_3]^T \in \mathbb{R}^4 \mid u_0 - 3u_1 + 3u_2 - u_3 = 0\}.$$

Exercise 10. Given vectors $[p, q]^T, [r, s]^T \in \mathbb{R}^2$, we can define a linear map $\phi\colon M_2\mathbb{R} \to \mathbb{R}$ by

$$\phi(A) = \begin{bmatrix} p & q \end{bmatrix} A \begin{bmatrix} r \\ s \end{bmatrix}.$$

Show that p, q, r and s **cannot** be chosen so that $\phi(A) = \operatorname{trace}(A)$ for all $A \in M_2\mathbb{R}$.