Vector Spaces and Fourier Theory — Problem Sheet 5

Please hand in questions

Exercise 1. Define $\phi\colon \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\phi \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y+z \\ z+x \\ x+y \end{bmatrix}.$$

Find the matrix of ϕ with respect to the usual basis of $\mathbb{R}^3$. Then find the matrix with respect to the basis

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad \mathbf{u}_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Exercise 2. Define a linear map $\phi\colon \mathbb{R}[x]_{\leq 2} \to \mathbb{R}^3$ by $\phi(f) = [f(0), f'(1), f''(2)]^T$. What is the matrix of ϕ with respect to the usual bases of $\mathbb{R}[x]_{\leq 2}$ and $\mathbb{R}^3$?

Exercise 3. Fix a real number λ, and let V be the set of functions of the form

$$f(x) = (ax^2 + bx + c)e^{\lambda x}.$$

In other words, we have $V = \mathbb{R}[x]_{\leq 2} e^{\lambda x}$.

(a) Write down a basis for V.
(b) Show that if $f \in V$ then $f' \in V$, so we can define a linear map $D\colon V \to V$ by $D(f) = f'$.
(c) What is the matrix of D with respect to your chosen basis?
(d) Show that $(D - \lambda)^3(f) = 0$ for all $f \in V$.

Exercise 4. Put $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in M_2\mathbb{R}$. Define $\phi\colon \mathbb{R}[x]_{\leq 2} \to M_2\mathbb{R}$ by $\phi(f) = f(J)$, or in other words

$$\phi(ax^2 + bx + c) = aJ^2 + bJ + cI.$$

Find bases for $\ker(\phi)$ and $\operatorname{image}(\phi)$.

Exercise 5. Let V and W be vector spaces, and let $\phi\colon V \to W$ be a linear map. Let $\mathcal{V} = v_1, \dots, v_n$ be a list of elements of $\mathcal{V}$.

(a) Show that if $v_1, \dots, v_n$ are linearly dependent, then so are $\phi(v_1), \dots, \phi(v_n)$.
(b) Give an example where $v_1, \dots, v_n$ are linearly independent, but $\phi(v_1), \dots, \phi(v_n)$ are linearly dependent.
(c) Show that if $\phi(v_1), \dots, \phi(v_n)$ are linearly independent, then $v_1, \dots, v_n$ are linearly independent.

Exercise 6. Let V be a vector space, and let $\phi\colon \mathbb{R}[x]_{\leq 2} \to V$ be a linear map. Show that there exist elements $u, v \in V$ such that

$$\phi(ax + b) = au + bv$$

for all $a, b \in \mathbb{R}$.

Exercise 7. Define subspaces $V, W \leq \mathbb{R}[x]_{\leq 3}$ by

$$V = \{f \in \mathbb{R}[x]_{\leq 3} \mid f(x) + f(-x) = 0\}$$
$$W = \{f \in \mathbb{R}[x]_{\leq 3} \mid f''(1) = 2f'(1) = 6f(1)\}$$

Find bases for V, W and $V \cap W$. Prove that

$$V + W = \{f \in \mathbb{R}[x]_{\leq 3} \mid f''(0) = 6f(0)\}.$$

Exercise 8. Define a map $\Delta\colon \mathbb{R}[x]_{\leq 4} \to \mathbb{R}[x]_{\leq 3}$ by $\Delta(f(x)) = f(x+1) - f(x)$. What is the matrix of this map with respect to the the usual bases of $\mathbb{R}[x]_{\leq 4}$ and $\mathbb{R}[x]_{\leq 3}$? What are the kernel and image of Δ?