Vector Spaces and Fourier Theory — Problem Sheet 6

There will be an online test this week, which covers all the questions on this sheet.

Exercise 1. Consider the vectors

$$\mathbf{a} = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \mathbf{u}_1 = \begin{bmatrix} 16 \\ -12 \\ 15 \end{bmatrix} \quad \mathbf{u}_2 = \begin{bmatrix} 15 \\ 20 \\ 0 \end{bmatrix} \quad \mathbf{u}_3 = \begin{bmatrix} 12 \\ -9 \\ -20 \end{bmatrix}$$

Define $\alpha\colon \mathbb{R}^3 \to \mathbb{R}^3$ by $\alpha(\mathbf{x}) = \mathbf{a} \times \mathbf{x}$. Let A be the matrix of α with respect to the basis $\mathcal{U} = \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

(a) Calculate $\alpha(\mathbf{u}_i)$ for $i = 1, 2, 3$, and observe that the answer is always a multiple of $\mathbf{u}_j$ for some j.
(b) Hence write down the matrix A.
(c) Calculate $\mu_{\mathcal{U}}(\mathbf{b})$, $\alpha(\mu_{\mathcal{U}}(\mathbf{b}))$, $\phi_A(\mathbf{b})$ and $\mu_{\mathcal{U}}(\phi_A(\mathbf{b}))$. Check that $\alpha(\mu_{\mathcal{U}}(\mathbf{b})) = \mu_{\mathcal{U}}(\phi_A(\mathbf{b}))$.

Exercise 2. Define maps $\alpha, \beta\colon M_2\mathbb{R} \to M_2\mathbb{R}$ by

$$\alpha(X) = X - X^T \qquad\qquad \beta(X) = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right] X.$$

Put $\mathcal{E} = E_1, E_2, E_3, E_4$, where

$$E_1 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right] \quad E_2 = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] \quad E_3 = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right] \quad E_4 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right].$$

Let A be the matrix of α with respect to the basis $\mathcal{E}$, let B be the matrix of β with respect to $\mathcal{E}$, and let C be the matrix of $\alpha\beta$ with respect to $\mathcal{E}$.

(a) Find $\alpha(E_i)$ for each i, and hence find A.
(b) Find $\beta(E_i)$ for each i, and hence find B.
(c) Find $\alpha(\beta(E_i))$ for each i, and hence find C.
(d) Check that $C = AB$.

Exercise 3. Let α, $\mathcal{E}$ and A be as in the previous exercise. Now consider the alternative basis $\mathcal{E}' = E'_1, E'_2, E'_3, E'_4$, where

$$E'_1 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right] \quad E'_2 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right] \quad E'_3 = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] \quad E'_4 = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right].$$

Let P be the change of basis matrix from $\mathcal{E}$ to $\mathcal{E}'$, and let A' be the matrix of α with respect to $\mathcal{E}'$.

(a) Express each matrix E'_i as a linear combination of $E_1, \dots, E_4$, and hence write down the matrix P.
(b) Express each matrix $\alpha(E'_i)$ as a linear combination of $E'_1, \dots, E'_4$, and hence write down the matrix A'.
(c) Check that $PA' = AP$.

Exercise 4. Define $\phi\colon \mathbb{R}[x]_{\leq 2} \to \mathbb{R}[x]_{\leq 2}$ by $\phi(f) = f + f' + f''$. Find the matrix of ϕ with respect to a suitable basis, and hence calculate $\mathrm{trace}(\phi)$, $\det(\phi)$ and $\mathrm{char}(\phi)(t)$.

Exercise 5. Consider the following elements of $\mathbb{R}^6$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{v}_5 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{v}_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{v}_7 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{v}_8 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ -1 \\ -1 \end{bmatrix}$$

Put $\mathcal{V} = \mathbf{v}_1, \dots, \mathbf{v}_8$ and $V_0 = 0$ and $V_j = \mathrm{span}(\mathbf{v}_1, \dots, \mathbf{v}_j)$ for $j > 0$. Recall that i is a *jump* for the sequence $\mathcal{V}$ if $v_i \notin V_{i-1}$. Find all the jumps.