Vector Spaces and Fourier Theory — Problem Sheet 7

You are not asked to hand in any questions this week.

Exercise 1. Let U be a finite-dimensional vector space, and let V and W be subspaces of U. In lectures we proved that there exist elements

$$u_1, \dots, u_p, v_1, \dots, v_q, w_1, \dots, w_r$$

such that

- $u_1, \dots, u_p$ is a basis for $V \cap W$
- $u_1, \dots, u_p, v_1, \dots, v_q$ is a basis for V
- $u_1, \dots, u_p, w_1, \dots, w_r$ is a basis for W
- $u_1, \dots, u_p, v_1, \dots, v_q, w_1, \dots, w_r$ is a basis for $V + W$.

Find elements as above for the case $U = M_2\mathbb{R}$ and $V = \{A \in U \mid A^T = A\}$ and $W = \{A \in U \mid \operatorname{trace}(A) = 0\}$.

Exercise 2. Let Z be a finite-dimensional vector space, and let U, V and W be subspaces of Z. Suppose that

$$\dim(U) = 2 \qquad \dim(U \cap V) = 1$$
$$\dim(V) = 3 \qquad \dim(V \cap W) = 2$$
$$\dim(W) = 4 \qquad \dim((U + V) \cap W) = 3.$$

Find the dimensions of $U + V$, $V + W$ and $U + V + W$. Hence show that $U + V + W = V + W$ and thus that $U \leq V + W$.

Exercise 3. Let $\phi\colon U \to V$ be a linear map between finite-dimensional vector spaces. Recall that there exists a number r and bases $u_1, \dots, u_n$ (for U) and $v_1, \dots, v_m$ (for V) such that

$$\phi(u_i) = \begin{cases} v_i & \text{if } i \leq r \\ 0 & \text{if } i > r. \end{cases}$$

(The method is to find a basis $v_1, \dots, v_r$ for $\operatorname{image}(\phi)$, choose elements $u_1, \dots, u_r$ with $\phi(u_i) = v_i$, then choose any basis $u_{r+1}, \dots, u_n$ for $\ker(\phi)$, then choose any elements $v_{r+1}, \dots, v_m$ for V such that $v_1, \dots, v_m$ is a basis for V.)

Find such adapted bases for the following maps:

(a) $\phi\colon M_2\mathbb{R} \to \mathbb{R}[x]_{\leq 3}$, $\phi(A) = [x, x^2] A \left[\begin{smallmatrix} 1 \\ x \end{smallmatrix}\right]$
(b) $\psi\colon \mathbb{R}[x]_{\leq 2} \to \mathbb{R}^3$, $\psi(f) = [f(1), f(-1), f'(0)]^T$
(c) $\chi\colon M_2\mathbb{R} \to M_3\mathbb{R}$, $\chi(A) = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & -\operatorname{trace}(A) \end{array}\right]$
(d) $\theta = \mu_{\mathcal{P}}\colon \mathbb{R}^4 \to \mathbb{R}[x]_{\leq 2}$, where $\mathcal{P} = x^2, (x+1)^2, (x-1)^2, x^2 + 1$.

Exercise 4. Let V be the set of all sequences $(a_0, a_1, a_2, \dots)$ for which $a_{i+2} = 3a_{i+1} - 2a_i$ for all i.

(a) Define $\pi\colon V \to \mathbb{R}^2$ by

$$\pi(a_0, a_1, a_2, a_3, \dots) = [a_0, a_1]^T.$$

Show that $\ker(\pi) = 0$, so π is injective.

(b) Define sequences u, v by $u_i = 1$ for all i, and $v_i = 2^i$. Show that $u, v \in V$.
(c) Find constants p, q, r, s such that the elements $b = pu + qv$ and $c = ru + sv$ satisfy $\pi(b) = [1, 0]^T$ and $\pi(c) = [0, 1]^T$.
(d) Show that b and c give a basis for V, and deduce that u and v give a basis for V.
(e) Define $\lambda\colon V \to V$ by

$$\lambda(a_0, a_1, a_2, a_3, \dots) = (a_1, a_2, a_3, a_4, \dots).$$

What is the matrix of λ with respect to the basis u, v?