Vector Spaces and Fourier Theory — Problem Sheet 8

There will be an online test covering parts of all the questions below.

Exercise 1. Use the usual inner product $\langle A,B\rangle = \mathrm{trace}(AB^T)$ on $M_3\mathbb{R}$.

(a) Calculate all the inner products $\langle C_i, C_j\rangle$, where

$$C_1 = \begin{bmatrix} 1&1&1\\0&1&1\\0&0&1\end{bmatrix} \qquad C_2 = \begin{bmatrix} 1&2&3\\2&1&2\\3&2&1\end{bmatrix} \qquad C_3 = \begin{bmatrix} 0&1&2\\-1&0&3\\-2&-3&0\end{bmatrix}$$

(b) Show that if $A^T = A$ and $B^T = -B$ then A and B are orthogonal.

(c) Put $\mathbf{u} = [1,1,1]^T$, and let V be the set of all matrices B such that the all three columns of B are the same. Show that if A is orthogonal to V then $A\mathbf{u} = 0$.

Exercise 2. Use the inner product $\langle f,g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$ on $\mathbb{R}[x]_{\leq 2}$.

(a) Find $\langle x+1, x^2+x\rangle$

(b) Show that if $0 \leq i,j \leq 2$ and $i+j$ is odd then $\langle x^i, x^j\rangle = 0$.

(c) Consider a polynomial $u(x) = px^2 + q$, and another polynomial $f(x) = ax^2 + bx + c$. Give a formula for $4f(-1) - 8f(0) + 4f(1)$ and another formula for $\langle f,u\rangle$. Hence find p and q such that $\langle f,u\rangle = 4f(-1) - 8f(0) + 4f(1)$ for all quadratic polynomials f.

Exercise 3. Show that for any $f \in C[-1,1]$ we have

$$\left|\int_{-1}^{1} \sqrt{1-x^2}\, f(x)\,dx\right| \leq \frac{2}{\sqrt{3}}\left(\int_{-1}^{1} f(x)^2\,dx\right)^{1/2}$$

Find a nonzero function $f \in C[-1,1]$ for which the above inequality is actually an equality.

Exercise 4. Show that for any $f \in C[0,1]$ we have

$$\left(\int_0^1 f(x)^3\,dx\right)^2 \leq \left(\int_0^1 f(x)^2\,dx\right)\left(\int_0^1 f(x)^4\,dx\right)$$

For which functions f is this actually an equality?

Exercise 5. Consider the space $U = M_2\mathbb{R}$ and the subspace $V = \{A \in U \mid A^T = A\}$. Given matrices $A,B \in U$, put

$$\langle A,B\rangle = \det(A-B) - \det(A+B) + 2\,\mathrm{trace}(A)\,\mathrm{trace}(B).$$

Expand out $\langle A,B\rangle$ when $A = \begin{bmatrix} a_1 & a_2\\ a_3 & a_4\end{bmatrix}$ and $B = \begin{bmatrix} b_1 & b_2\\ b_3 & b_4\end{bmatrix}$. Show that

(a) $\langle A+B, C\rangle = \langle A,C\rangle + \langle B,C\rangle$ for all $A,B,C \in U$.

(b) $\langle tA, B\rangle = t\langle A,B\rangle$ for all $A,B \in U$ and $t \in \mathbb{R}$.

(c) $\langle A,B\rangle = \langle B,A\rangle$ for all $A,B \in U$.

(d) There exists $A \in U$ such that $\langle A,A\rangle < 0$.

(e) However, if $A \in V$ then $\langle A,A\rangle \geq 0$, with equality iff $A = 0$.

Exercise 6. Put

$$V = \{f \in C^\infty(\mathbb{R}) \mid f + f'' = 0\}.$$

For $f,g \in V$ put

$$\langle f,g\rangle(t) = f(t)g(t) + f'(t)g'(t),$$

so $\langle f,g\rangle \in C^\infty(\mathbb{R})$.

(a) Prove that $\langle f,g\rangle$ is actually a constant.

(b) Prove that if $f \in V$ then $f' \in V$, so that differentiation gives a linear map $D\colon V \to V$.

(c) The functions sin and cos give a basis for V. Using this, show that $\langle , \rangle$ is an inner product on V.

(d) What is the matrix of D with respect to the basis $\{\sin, \cos\}$?

Exercise 7. Put $V = \{B \in M_2\mathbb{R} \mid B^T = B\}$, and let $\pi\colon M_2\mathbb{R} \to V$ be the orthogonal projection. Find an orthogonal basis for V, and use it to calculate $\pi(A)$ for an arbitrary matrix $A = \begin{bmatrix} a & b\\ c & d\end{bmatrix}$. Use this to show that $\pi(A) = (A + A^T)/2$.