Vector Spaces and Fourier Theory — Problem Sheet 9

Please hand in answers to exercises 1 and 5. Because of the Bank Holiday on Monday May 5th, I will ask you to put your work in the folder pinned to the door of my office (J26) by 12.00 on May 6th. I will try to get it marked in time for tutorials on May 7th.

Exercise 1. Let $\mathcal{W} = w_1, \dots, w_p$ be a strictly orthogonal sequence in an inner product space V, and let v be an element of V. Show that

$$\|v\|^2 \geq \sum_{i=1}^{p} \frac{\langle v, w_i\rangle^2}{\langle w_i, w_i\rangle}.$$

(Note that Parseval's inequality covers the case where the sequence is orthonormal, so $\|w_i\| = 1$. You can prove the above statement either by modifying the proof of Parseval's inequality, or by applying Parseval's inequality to a different sequence.)

Exercise 2. Use the result in Exercise 1 to show that for any continuous function $f \in C[-1, 1]$ we have

$$2 \int_{-1}^{1} f(x)^2\,dx \geq \left(\int_{-1}^{1} f(x)\,dx\right)^2 + 3\left(\int_{-1}^{1} x\,f(x)\,dx\right)^2.$$

Exercise 3. Consider the space $V = M_4\mathbb{R}$ with the usual inner product $\langle A, B\rangle = \mathrm{trace}(AB^T)$. Consider the following sequence in V:

$$A_1 = \begin{bmatrix} 0&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&0 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 0&0&0&0\\ 0&1&1&0\\ 0&1&1&0\\ 0&0&0&0 \end{bmatrix} \qquad A_3 = \begin{bmatrix} 0&1&1&0\\ 1&1&1&1\\ 1&1&1&1\\ 0&1&1&0 \end{bmatrix} \qquad A_4 = \begin{bmatrix} 1&1&1&1\\ 1&1&1&1\\ 1&1&1&1\\ 1&1&1&1 \end{bmatrix}$$

Find an orthonormal sequence $C_1, \dots, C_4$ in V such that $\mathrm{span}\{A_1, \dots, A_i\} = \mathrm{span}\{C_1, \dots, C_i\}$ for all i. (You can use the Gram-Schmidt procedure for this but it is easier to find an answer by inspection.)

Exercise 4. Consider the following vectors in $\mathbb{R}^5$:

$$u_1 = \begin{bmatrix} 1\\1\\1\\1\\1 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 1\\1\\1\\1\\0 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 1\\1\\1\\0\\0 \end{bmatrix} \qquad u_4 = \begin{bmatrix} 1\\1\\0\\0\\0 \end{bmatrix} \qquad u_5 = \begin{bmatrix} 1\\0\\0\\0\\0 \end{bmatrix}$$

Find an orthonormal sequence $\hat{v}_1, \dots, \hat{v}_5$ such that $\mathrm{span}\{\hat{v}_1, \dots, \hat{v}_i\} = \mathrm{span}\{u_1, \dots, u_i\}$ for all i.

Exercise 5. Define an inner product on $\mathbb{R}[t]_{\leq 2}$ by

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(t)g(t)e^{-t^2}\,dt/\sqrt{\pi}.$$

Apply the Gram-Schmidt procedure to the basis $\{1, t, t^2\}$ to get a basis for $\mathbb{R}[t]_{\leq 2}$ that is orthonormal with respect to this inner product. You may assume that

$$\begin{aligned} \langle t^n, t^m\rangle &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} t^{n+m} e^{-t^2}\,dt \\ &= \begin{cases} \frac{1}{2^{n+m}} \frac{(n+m)!}{((n+m)/2)!} & \text{if } n+m \text{ is even} \\ 0 & \text{if } n+m \text{ is odd} \end{cases} \end{aligned}$$

(and you should remember that $0! = 1$).

Exercise 6. For $x \in \mathbb{R}^4$, put

$$\alpha(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2 - \frac{(x_1 - x_2)^2}{2} - \frac{(x_3 - x_4)^2}{2} - \frac{(x_1 + x_2 + x_3 + x_4)^2}{4}$$

(a) By finding a suitable orthonormal sequence v_1, v_2, v_3, show that $\alpha(x) \geq 0$ for all $x \in \mathbb{R}^4$.
(b) Find a fourth vector v_4 such that v_1, v_2, v_3, v_4 is orthonormal.
(c) Expand out and simplify $\alpha(x)$. How is the answer related to (b)?