Vector Spaces and Fourier Theory — Problem Sheet 10

Everywhere on this sheet we use the standard inner products on $\mathbb{R}^n$, $M_n\mathbb{R}$ and T_n, given by $\langle \mathbf{v}, \mathbf{w}\rangle = \sum_{i=1}^n v_i w_i$ and $\langle A, B\rangle = \operatorname{trace}(AB^T) = \operatorname{trace}(A^T B)$ and $\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{g(t)}\,dt$ respectively.

There will be an online test covering parts of Exercises 1 to 8.

Exercise 1. Consider the map $\phi\colon \mathbb{R}^3 \to M_2\mathbb{R}$ given by $\phi\left[\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right] = \left[\begin{smallmatrix} x & y \\ y & z \end{smallmatrix}\right]$. Given a matrix $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, find a vector $\mathbf{w} = [p, q, r]^T$ such that $\langle \phi(\mathbf{v}), A\rangle = \langle \mathbf{v}, \mathbf{w}\rangle$ for all vectors $\mathbf{v} \in \mathbb{R}^3$. (The adjoint map $\phi^*\colon M_2\mathbb{R} \to \mathbb{R}^3$ is then given by $\phi^*(A) = \mathbf{w}$.)

Exercise 2. Consider the map $\phi\colon M_3\mathbb{R} \to M_3\mathbb{R}$ given by

$$\phi\left[\begin{smallmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & a_4 & a_7 \\ 0 & 0 & a_8 \\ 0 & 0 & 0 \end{smallmatrix}\right].$$

Give a formula for the adjoint map $\phi^*\colon M_3\mathbb{R} \to M_3\mathbb{R}$.

Exercise 3. Consider the map $\phi\colon M_2\mathbb{R} \to M_2\mathbb{R}$ given by $\phi(A) = QAQ$, where $Q = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$. Show that $\phi^* = \phi$.

Exercise 4. Define $\phi\colon M_n\mathbb{R} \to M_n\mathbb{R}$ by $\phi(A) = A - \frac{1}{n}\operatorname{trace}(A)I$. Show that $\phi^* = \phi$.

Exercise 5. In this exercise we give the space $\mathbb{R}[x]_{\leq 2}$ the inner product $\langle f, g\rangle = \int_{-1/2}^{1/2} f(x)g(x)\,dx$. Define $\chi\colon \mathbb{R}[x]_{\leq 2} \to \mathbb{R}$ by $\chi(f) = f''(0)$. If $f(x) = ax^2 + bx + c$, what is $\chi(f)$? Find an element $u \in \mathbb{R}[x]_{\leq 2}$ such that $\chi(f) = \langle f, u\rangle$ for all f, and thus give a formula for χ^*.

Exercise 6. Let T_2 be the usual space of trigonometric polynomials. We can define $\Delta\colon T_2 \to T_2$ by $\Delta(f) = f''$.

(a) Find $\Delta(f)$, where $f = \sum_{n=-2}^{2} a_n e_n$.
(b) Show that Δ is self-adjoint. (This can be deduced from part (a), or you can prove it more directly.)
(c) Find the eigenvalues of Δ (there are three of them).
(d) What are the dimensions of the corresponding eigenspaces?

Exercise 7. Suppose that $f \in T_2$ satisfies $f(0) = f(\pi)$ and $f(-\pi/2) = f(\pi/2)$. Show that $f(t + \pi) = f(t)$ for all t.

Exercise 8. Put $U = \{f \in T_2 \mid f(0) = f'(0) = 0\}$. The aim of this exercise is to find an orthonormal basis for $U^\perp$.

(a) If $f = \sum_{n=-2}^{2} a_n e_n$, find $f(0)$ and $f'(0)$ in terms of the numbers a_n, and so find the general form of an element of U.
(b) Using this, find a basis for U.
(c) Using this and the fact that $T_2 = U \oplus U^\perp$, find the dimensions of U and $U^\perp$.
(d) Using part (a) again, find elements $v_0, v_1 \in T_2$ such that $\langle f, v_0\rangle = f(0)$ and $\langle f, v_1\rangle = f'(0)$ for all $f \in T_2$.
(e) Show that v_0, v_1 is an orthogonal basis for $U^\perp$, and thus find an orthonormal basis.

Exercise 9. Let U and V be vector spaces with inner products, and let $\phi\colon U \to V$ be a linear map with the property that $\phi^*(\phi(u)) = u$ for all $u \in U$. Let $\mathcal{U} = u_1, \dots, u_n$ be an orthonormal sequence in U. Show that $\phi(u_1), \dots, \phi(u_n)$ is an orthonormal sequence in V.

Exercise 10. Let U and V be vector spaces with inner products, and let $\phi\colon U \to V$ be a linear map with the property that $\phi(\phi^*(v)) = v$ for all $v \in V$. Let u be an element of U, and put $u_1 = \phi^*\phi(u)$ and $u_2 = u - u_1$.

(a) Show that $\phi(u_1) = \phi(u)$ and $\phi(u_2) = 0$.
(b) Show that $\langle u_1, u_2\rangle = 0$.
(c) Deduce that $\|u\|^2 \geq \|u_1\|^2$
(d) Show that $\|u_1\|^2 = \|\phi(u)\|^2$.
(e) Deduce that $\|\phi(u)\| \leq \|u\|$.