Vector Spaces and Fourier Theory — Problem Sheet 11

This week there is nothing to hand in, and no online test.

Exercise 1. Define $\alpha\colon \mathbb{C}^3 \to \mathbb{C}^3$ by $\alpha\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+2y+z \\ 2x+9y+2z \\ x+2y+z \end{bmatrix}$.

(a) What is the matrix of α with respect to the standard basis of $\mathbb{C}^3$?
(b) Show that α is self-adjoint.
(c) Find an orthonormal basis of $\mathbb{C}^3$ consisting of eigenvectors of α.

Exercise 2. Define $\alpha\colon \mathbb{C}^5 \to \mathbb{C}^5$ by

$$\alpha([z_0, z_1, z_2, z_3, z_4]^T) = [z_1, z_2, z_3, z_4, z_0]^T.$$

(a) Find $\alpha^\dagger$, and show that $\alpha^\dagger = \alpha^{-1}$. (Of course this is a special property of this particular map. For most linear maps, the adjoint is unrelated to the inverse.)
(b) Find the eigenvalues of α. (Hint: it is easier to think directly about when $\alpha(z) = \lambda\, z$, rather than trying to calculate the characteristic polynomial. You will need to consider separately the cases where $\lambda^5 = 1$ and where $\lambda^5 \neq 1$.)

Exercise 3. Define a map $\alpha\colon \mathbb{R}[x]_{\leq 2} \to \mathbb{R}[x]_{\leq 2}$ by $\alpha(f) = (3x^2 - 1)f''$. You may assume that if we use the inner product $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$ on $\mathbb{R}[x]_{\leq 2}$, then α is self-adjoint.

(a) Show that $\alpha(\alpha(f)) = 6\alpha(f)$ for all f.
(b) Deduce that if f is a nonzero eigenvector of α with eigenvalue λ, then $\alpha(\alpha(f)) = \lambda^2 f$ and $\lambda^2 = 6\lambda$.
(c) Find an orthogonal basis for $\mathbb{R}[x]_{\leq 2}$ consisting of eigenvectors for α.

Exercise 4. Let V be the space of functions $f\colon \mathbb{R} \to \mathbb{C}$ of the form $f(x) = p(x)e^{-x^2/2}$ for some $p \in \mathbb{C}[x]$. Give this the Hermitian form $\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx$. Define $\phi\colon V \to V$ by $\phi(f)(x) = x\,f(x)$.

(a) Show that ϕ is self-adjoint.
(b) Show that if $\phi(f) = \lambda f$ for some $\lambda \in \mathbb{C}$, then $f = 0$. (Thus, ϕ has no eigenvalues. This is only possible because V is infinite-dimensional.)

Exercise 5. Let T be the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and define $\gamma\colon M_2\mathbb{R} \to M_2\mathbb{R}$ by $\gamma(A) = TA - AT$.

(a) Give a basis for $M_2\mathbb{R}$, and find the matrix of γ with respect to that basis.
(b) Find bases for the kernel and the image of γ. Show that the image is the orthogonal complement of the kernel with respect to the usual inner product $\langle X, Y\rangle = \mathrm{trace}(XY^T)$ on $M_2\mathbb{R}$.
(c) Show that $\gamma^4 = 4\gamma^2$.
(d) Find a basis of $M_2\mathbb{R}$ consisting of eigenvectors for γ. (Note here that an eigenvector for γ is a *matrix* A such that $\gamma(A) = TA - AT = \lambda A$ for some λ.)