Vector Spaces and Fourier Theory — Problem Sheet 1

Solution 1. Of course there are many different correct answers to this question. The following will do:

(a) $u = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $v = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$, $u + v = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}$, $10v = \begin{bmatrix} 10 \\ -10 \\ 10 \\ -10 \end{bmatrix}$.

(b) $u = \left[\begin{smallmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{smallmatrix}\right]$, $v = \left[\begin{smallmatrix} 6 & 5 & 4 \\ 3 & 2 & 1 \end{smallmatrix}\right]$, $u + v = \left[\begin{smallmatrix} 7 & 7 & 7 \\ 7 & 7 & 7 \end{smallmatrix}\right]$, $10v = \left[\begin{smallmatrix} 60 & 50 & 40 \\ 30 & 20 & 10 \end{smallmatrix}\right]$.

(c) $u = 1 + x$, $v = x + x^2$, $u + v = 1 + 2x + x^2$, $10v = 10x + 10x^2$.

(d) u is the vector pointing 10 miles east, v is the vector pointing 20 miles west, $u + v$ points ten miles west, $10v$ points 200 miles west.

Solution 2. (a) This is not a vector space because $\left[\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right] \in V_0$ but $(-1).\left[\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right] = \left[\begin{smallmatrix} -1 & -2 \\ -3 & -4 \end{smallmatrix}\right] \not\in V_0$, which contradicts axiom (b) of Predefinition 2.1.

(b) This is not a vector space because the zero matrix does not lie in V_1.

(c) This is not a vector space because $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] \in V_2$ and $\left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right] \in V_2$ but $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] + \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}\right] \not\in V_2$, which contradicts axiom (a).

(d) To see that this is not a vector space, consider the polynomials $p(x) = x$ and $q(x) = 1 - x$. Then $p(0)p(1) = 0 = q(0)q(1)$, so $p \in V_3$ and $q \in V_3$. However, $p(x) + q(x) = 1$ for all x, so $p + q \not\in V_3$. This contradicts axiom (a).

Solution 3. (a) These are linearly dependent, and they do not span. Indeed, any list of four vectors in $\mathbb{R}^3$ is always dependent. Explicitly, we have $\mathbf{u}_1 - \mathbf{u}_2 - \mathbf{u}_3 + \mathbf{u}_4 = 0$, which gives a direct proof of dependence. Also, all the vectors $\mathbf{u}_i$ have zero as the second entry, so the same will be true for any vector in the span of the vectors $\mathbf{u}_i$. In particular, the vector $[0, 1, 0]^T$ does not lie in that span, so the $\mathbf{u}_i$'s do not span all of $\mathbb{R}^3$. This means that they do not form a basis.

(b) Any list of four vectors in $\mathbb{R}^3$ is automatically linearly dependent (and so cannot form a basis). More specifically, the relation $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 - 2\mathbf{v}_4 = 0$ shows that the $\mathbf{v}_i$'s are dependent. These vectors span all of $\mathbb{R}^3$, because any vector $\mathbf{a} = \left[\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right] \in \mathbb{R}^3$ can be expressed as $\mathbf{a} = -z\mathbf{v}_1 - y\mathbf{v}_2 - x\mathbf{v}_3 + (x + y + z)\mathbf{v}_4$.

(c) A list of two vectors can only be linearly dependent if one is a multiple of the other, which is clearly not the case here, so $\mathbf{w}_1$ and $\mathbf{w}_2$ are linearly independent. Moreover, a list of two vectors can never span all of $\mathbb{R}^3$. More explicitly, we claim that the vector $\mathbf{e}_1 = \left[\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right]$ cannot be expressed as a linear combination of $\mathbf{w}_1$ and $\mathbf{w}_2$. Indeed, if we have $\lambda_1\mathbf{w}_1 + \lambda_2\mathbf{w}_2 = \mathbf{e}_1$ then

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \lambda_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} \lambda_1 + 4\lambda_2 \\ 2\lambda_1 + 5\lambda_2 \\ 3\lambda_1 + 6\lambda_2 \end{bmatrix},$$

so

$$\lambda_1 + 4\lambda_2 = 0 \qquad 2\lambda_1 + 5\lambda_2 = 1 \qquad 3\lambda_1 + 6\lambda_2 = 0.$$

The first and third of these easily give $\lambda_1 = \lambda_2 = 0$, which is incompatible with the second equation, so there is no solution. This shows that $\mathbf{w}_1$ and $\mathbf{w}_2$ do not form a basis of $\mathbb{R}^3$.

(d) The vectors $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ are linearly independent and span $\mathbb{R}^3$, so they form a basis. One way to see this is to write down the matrix $A = \left[\begin{smallmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 4 \end{smallmatrix}\right]$ whose columns are $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$, and observe that it row-reduces almost instantly to the identity. Alternatively, we must show that for any vector $\mathbf{a} = \left[\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right] \in \mathbb{R}^3$, there are unique real numbers λ, μ, ν such that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} + \nu \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}.$$

This equation is equivalent to $\lambda = x$ and $\lambda + 2\mu = y$ and $\lambda + 2\mu + 4\nu = z$. It is easy to see that there is indeed a unique solution, namely $\lambda = x$ and $\mu = (y - x)/2$ and $\nu = (z - y)/4$.

Solution 4. (a) $U \cap V$ is the set of vectors $[w, x, y, z]^T$ satisfying the three equations

$$\begin{aligned} w - x + y - z &= 0 \\ w + x + y &= 0 \\ x + y + z &= 0. \end{aligned}$$

Subtracting the last two equations gives $w = z$. Putting this back into the first equation gives $x = y$. The middle equation now gives $w = -2x$, so

$$[w, x, y, z] = [-2x, x, x, -2x].$$

Thus

$$U \cap V = \{[-2x, x, x, -2x]^T \mid x \in \mathbb{R}\} = \operatorname{span}([-2, 1, 1, -2]^T).$$

(b) $U \cap W$ is the set of vectors of the form $[u, u + v, u + 2v, u + 3v]^T$ for which $u - (u + v) + (u + 2v) - (u + 3v) = 0$, which reduces to $-2v = 0$, or equivalently $v = 0$. Thus

$$U \cap W = \{[u, u, u, u]^T \mid u \in \mathbb{R}\} = \operatorname{span}([1, 1, 1, 1]^T)$$

(c) $V \cap W$ is the set of vectors of the form $[u, u + v, u + 2v, u + 3v]^T$ for which $u + (u + v) + (u + 2v) = 0 = (u + v) + (u + 2v) + (u + 3v)$, or in other words $3u + 3v = 0 = 3u + 6v$. These equations easily imply that $u = v = 0$, and this means that $V \cap W = 0$.

Solution 5. If you choose two planes at random, their intersection will be a line (unless the two planes happened to be the same, which is unlikely). If you intersect this with a third randomly chosen plane, then you will just get the origin (barring unlikely coincidences). The special feature of P, Q and R is that $P \cap Q \cap R$ is not just the origin, but a line. Specifically, we have

$$P \cap Q \cap R = \left\{ \left[\begin{smallmatrix} t \\ -2t \\ t \end{smallmatrix} \right] \mid t \in \mathbb{R} \right\}.$$

Solution 6. The characteristic polynomial is

$$\begin{aligned} \det(tI - A) &= \det \left[\begin{smallmatrix} t & -a & -b \\ a & t & -c \\ b & c & t \end{smallmatrix} \right] = t \det \left[\begin{smallmatrix} t & -c \\ c & t \end{smallmatrix} \right] + a \det \left[\begin{smallmatrix} a & -c \\ b & t \end{smallmatrix} \right] - b \det \left[\begin{smallmatrix} a & t \\ b & c \end{smallmatrix} \right] \\ &= t(t^2 + c^2) + a(at + bc) - b(ac - bt) = t^3 + c^2 t + a^2 t + abc - abc + b^2 t \\ &= t^3 + (a^2 + b^2 + c^2)t. \end{aligned}$$

The eigenvalues of A are the roots of this polynomial. These are 0 (which is real) and $\pm i\sqrt{a^2 + b^2 + c^2}$ (which are purely imaginary).

Solution 7. The matrix can be row-reduced as follows:

$$\left[\begin{smallmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{smallmatrix} \right] \xrightarrow{1} \left[\begin{smallmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -2 \\ 0 & -2 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{smallmatrix} \right] \xrightarrow{2} \left[\begin{smallmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & -4 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{smallmatrix} \right] \xrightarrow{3} \left[\begin{smallmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{smallmatrix} \right] \xrightarrow{4} \left[\begin{smallmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{smallmatrix} \right] \xrightarrow{5} \left[\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{smallmatrix} \right] \xrightarrow{6} \left[\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix} \right].$$

(In step 1 we subtract row 1 from rows 2 and 3; in step 2 we subtract suitable multiples of row 4 from rows 1, 2, and 3; in step 3 we divide rows 2 and 3 by -4 and -2 respectively; in steps 4 and 5 we clear the 4th and 2nd columns; in step 6 we reorder the rows.) As the final matrix is the identity, we see that A is invertible and has rank 4.