Vector Spaces and Fourier Theory — Problem Sheet 2

Solution 1. The maps ϕ_0 and ϕ_3 are linear. Indeed, we have

$$\begin{aligned}
\phi_0\left(\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]+\left[\begin{smallmatrix}x'\\y'\end{smallmatrix}\right]\right) &= \phi_0\left[\begin{smallmatrix}x+x'\\y+y'\end{smallmatrix}\right] = \left[\begin{smallmatrix}(x+x')+(y+y')\\(x+x')-(y+y')\end{smallmatrix}\right]\\
&= \left[\begin{smallmatrix}x+y+x'+y'\\x-y+x'-y'\end{smallmatrix}\right] = \left[\begin{smallmatrix}x+y\\x-y\end{smallmatrix}\right]+\left[\begin{smallmatrix}x'+y'\\x'-y'\end{smallmatrix}\right]\\
&= \phi_0\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]+\phi_0\left[\begin{smallmatrix}x'\\y'\end{smallmatrix}\right]\\
\phi_0\left(t\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]\right) &= \phi_0\left[\begin{smallmatrix}tx\\ty\end{smallmatrix}\right] = \left[\begin{smallmatrix}tx+ty\\tx-ty\end{smallmatrix}\right] = t\left[\begin{smallmatrix}x+y\\x-y\end{smallmatrix}\right] = t\phi_0\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]\\
\phi_3(f+g) &= (f+g)(0)+(f+g)'(1)+(f+g)''(2) = f(0)+g(0)+f'(1)+g'(1)+f''(2)+g''(2)\\
&= (f(0)+f'(1)+f''(2))+(g(0)+g'(1)+g''(2)) = \phi_3(f)+\phi_3(g)\\
\phi_3(tf) &= (tf)(0)+(tf)'(1)+(tf)''(2) = t(f(0)+f'(1)+f''(2)) = t\phi_3(f).
\end{aligned}$$

For the others:

(b) Consider the vectors

$$\mathbf{e}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix} \qquad \mathbf{e}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix} \qquad \mathbf{e}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}.$$

Then $\phi_1(\mathbf{e}_1) = \phi_1(\mathbf{e}_2) = \phi_1(\mathbf{e}_3) = 0$. However, we have

$$\phi_1(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3) = \phi_1\begin{bmatrix}1\\1\\1\end{bmatrix} = 1,$$

so

$$\phi_1(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3) \neq \phi_1(\mathbf{e}_1)+\phi(\mathbf{e}_2)+\phi(\mathbf{e}_3),$$

so ϕ_1 is not linear.

(c) We have $\phi_2(I) = 1$ and $\phi_2((-1).I) = 1$, so $\phi_2((-1).I) \neq (-1).\phi_2(I)$, so ϕ_2 is not linear.

(e) Consider the polynomials $p(x) = x$ and $q(x) = 1-x$. Then $\phi_4(p) = 0\times 1 = 0$ and $\phi_4(q) = 1\times 0 = 0$, but $\phi_4(p+q) = 1\times 1 = 1$, so $\phi_4(p+q) \neq \phi_4(p)+\phi_4(q)$, so ϕ_4 is not linear.

Solution 2. Of course there are many different correct answers for this question. The following will do:

(a) $\phi\begin{bmatrix}w\\x\\y\\z\end{bmatrix} = \begin{bmatrix}w+x\\y+z\end{bmatrix}$

(b) $\phi\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix} = \begin{bmatrix}a\\i\end{bmatrix}$

(c) $\phi\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix} = ax^2+bx+c$

(d) $\phi(f) = \begin{bmatrix}f(0)&0\\0&f(1)\end{bmatrix}$.

Solution 3. No, because $\chi(0) = t^n \neq 0$, for example.

Solution 4. No. The eigenvalues of λI are all equal to λ, so $\rho(\lambda I) = |\lambda|$, whereas if ρ were linear we would have to have $\rho(\lambda I) = \lambda\rho(I) = \lambda$. Alternatively, we have $\rho(I) = \rho(-I) = 1$ but $\rho(0) = 0$, so $\rho(I+(-I)) \neq \rho(I)+\rho(-I)$.

Solution 5. (0) The set U_0 is a subspace. Indeed, it certainly contains the zero vector. If $[w,x,y,z]^T$ and $[w',x',y',z']^T$ lie in U_0, then $w+x = 0$ and $w'+x' = 0$, so $(w+w')+(x+x') = 0$, so the vector

$$[w,x,y,z]^T+[w',x',y',z']^T = [w+w',x+x',y+y',z+z']^T$$

also lies in U_0, so U_0 is closed under addition. If we also have $t\in\mathbb{R}$ then $tw+tx = t(w+x) = 0$, so $[tw,tx,ty,tz]^T \in U_0$, so U_0 is closed under scalar multiplication, and so is a subspace.

(1) The set U_1 is not a subspace, because it does not contain the zero vector.

(2) The set U_2 is a subspace. Indeed, it certainly contains the zero vector. If $[w,x,y,z]^T$ and $[w',x',y',z']^T$ lie in U_2, then $w+2x+3y+4z=0$ and $w'+2x'+3y'+4z'=0$, so

$$(w+w')+2(x+x')+3(y+y')+4(z+z')=(w+2x+3y+4z)+(w'+2x'+3y'+4z')=0+0=0,$$

so the vector

$$[w,x,y,z]^T+[w',x',y',z']^T=[w+w',x+x',y+y',z+z']^T$$

also lies in U_2, so U_2 is closed under addition. If we also have $t\in\mathbb{R}$ then $tw+2tx+3ty+4tz=t(w+2x+3y+4z)=0$, so $[tw,tx,ty,tz]^T\in U_2$, so U_2 is closed under scalar multiplication, and so is a subspace.

(3) The vector $[1,0,-1,0]^T$ lies in U_3, because $1+0^2+(-1)^3+0^4=0$. However, the vector $2.[1,0,-1,0]^T=[2,0,-2,0]^T$ does not lie in U_3, because $2+0^2+(-2)^3+0^4=-6\neq 0$. This shows that U_3 is not closed under scalar multiplication, so it is not a subspace.

(4) As w and x are real numbers, we have $w^2,x^2\geq 0$, so the only way we can have $w^2+x^2=0$ is if $w=x=0$. Thus

$$U_4=\{[w,x,y,z]^T\in\mathbb{R}^4 \mid w=x=0\}=\{[0,0,y,z]^T \mid y,z\in\mathbb{R}\}.$$

This is clearly a subspace of $\mathbb{R}^4$.

Solution 6. The set U_1 is not a subspace, because the zero function is not in U_1. The set U_2 is not a subspace either. Indeed, the constant function $f(t)=1$ is an element of U_2, but $(-1).f$ is not an element of U_2, so U_2 is not closed under scalar multiplication. The set U_4 is also not a subspace. To see this, consider the functions $f(x)=x(x-2)$ and $g(x)=(x-1)(x-2)$ and $h(x)=f(x)+g(x)=(2x-1)(x-2)$. Then

$$\begin{aligned} f(0)f(1)&=0=f(2)f(3)\\ g(0)g(1)&=0=g(2)g(3)\\ h(0)h(1)&=-2\neq h(2)h(3)=0. \end{aligned}$$

Thus $f,g\in U_4$ but $f+g\notin U_4$, so U_4 is not a subspace. However, U_0 and U_3 are subspaces of F.

Solution 7. Of course there are many different correct answers for this question. The following will do:

(a) $W=\{p\in\mathbb{R}[x]_{\leq 2} \mid p(0)=0\}=\{ax^2+bx \mid a,b\in\mathbb{R}\}$.

(b) $W=\{A\in M_{2,3}\mathbb{R} \mid A\begin{bmatrix}1\\0\\0\end{bmatrix}=0\}=\{\begin{bmatrix}0&a&b\\0&c&d\end{bmatrix} \mid a,b,c,d\in\mathbb{R}\}$

(c) $W=\{[x,y,z]^T\in V \mid z=0\}=\{[x,-x,0]^T \mid x\in\mathbb{R}\}$

Solution 8. Of course there are many different correct answers for this question. The following will do:

(a) $V=\{[w,x,0,0]^T \mid w,x\in\mathbb{R}\}$ and $W=\{[0,0,y,z]^T \mid w,x\in\mathbb{R}\}$.

(b) $V=\{\begin{bmatrix}w&x\\0&0\end{bmatrix} \mid w,x\in\mathbb{R}\}$ and $W=\{\begin{bmatrix}0&0\\y&z\end{bmatrix} \mid y,z\in\mathbb{R}\}$.

(c) $V=\{[s,-s,0]^T \mid s\in\mathbb{R}\}$ and $W=\{[0,t,-t]^T \mid t\in\mathbb{R}\}$.