Vector Spaces and Fourier Theory — Problem Sheet 3

Solution 1. Consider a polynomial $f \in U$, say $f(x) = ax^2 + bx + c$. We have $f(0) = c$, so $f \in V$ iff $c = 0$ iff $f(x) = ax^2 + bx$ for some $a, b \in \mathbb{R}$. On the other hand, we have

$$f(1) + f(-1) = (a + b + c) + (a - b + c) = 2(a + c),$$

so $f \in W$ iff $c = -a$, so $f(x) = a(x^2 - 1) + bx$ for some $a, b \in \mathbb{R}$. Thus $f \in V \cap W$ iff $c = 0$ and also $c = -a$, which means that $a = c = 0$, so $f(x) = bx$ for some b. This shows that $V \cap W$ is as claimed.

On the other hand, given an arbitrary quadratic polynomial $f(x) = ax^2 + bx + c$ we can put $g(x) = (a + c)x^2$ and $h(x) = bx - c(x^2 - 1)$. We then have $g(0) = 0$ and $h(1) + h(-1) = 0$, so $g \in V$ and $h \in W$, and $f = g + h$. This shows that $V + W = U$.

Aside: How did we find this g and h? We need g to be an element of V, so g must have the form $g(x) = px^2 + qx$ for some p, q. We also need h to be an element of W, so h must have the form $h(x) = r(x^2 - 1) + sx$ for some r, s. Finally, we need $f = g + h$, which means that

$$ax^2 + bx + c = (px^2 + qx) + (r(x^2 - 1) + sx) = (p + r)x^2 + (q + s)x - r.$$

By comparing coefficients we see that $a = p + r$ and $b = q + s$ and $c = -r$, so $r = -c$ and $p = a - r = a + c$ and $s = b - q$ with q arbitrary. We can choose to take $q = 0$, giving $s = b$ and so $g(x) = px^2 + qx = (a + c)x^2$ and $h(x) = r(x^2 - 1) + sx = -c(x^2 - 1) + bx$ as before.

Solution 2. Firstly, it is clear that $\alpha \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$ can only be zero if $u = v = 0$, so $\ker(\alpha) = 0$, so α is injective. Next, if $A \in \text{image}(\alpha)$ then $A = \left[\begin{smallmatrix} u & -u \\ v & -v \end{smallmatrix}\right]$ for some u and v, so $A \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} u & -u \\ v & -v \end{smallmatrix}\right] \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$. Conversely, suppose we have a matrix A with $A \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$. We can write $A = \left[\begin{smallmatrix} u & s \\ t & v \end{smallmatrix}\right]$ for some u, s, t and v, and we must have

$$\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} u & s \\ t & v \end{smallmatrix}\right] \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} u+s \\ t+v \end{smallmatrix}\right].$$

This means that $s = -u$ and $t = -v$, so $A = \left[\begin{smallmatrix} u & -u \\ -v & v \end{smallmatrix}\right] = \alpha \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$, so $A \in \text{image}(\alpha)$. This proves that $\text{image}(\alpha) = \{A \mid A \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] = 0\}$, as claimed.

Solution 3. (a) $\phi \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] - \dfrac{a + d}{2} \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} (a-d)/2 & b \\ c & (d-a)/2 \end{smallmatrix}\right]$

(b) We have $\text{trace}(I) = 2$, so $\phi(I) = I - \frac{1}{2}.2I = 0$, so $aI \in \ker(\phi)$ for all a. On the other hand, if $A \in \ker(\phi)$ then $A - \frac{1}{2}\,\text{trace}(A)I = 0$, so $A = \frac{1}{2}\,\text{trace}(A)I$, which is a multiple of I. Alternatively, we can write A as $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, and then if $\phi(A) = 0$ then part (a) gives $a - d = b = c = 0$, so $A = \left[\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right] = aI$. Either way, this completes the proof that $\ker(\phi) = \{aI \mid a \in \mathbb{R}\}$.

(c) For any matrix $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, we have

$$\text{trace}(\phi(A)) = \text{trace}\left(\left[\begin{smallmatrix} (a-d)/2 & b \\ c & (d-a)/2 \end{smallmatrix}\right] \right) = \frac{a - d}{2} + \frac{d - a}{2} = 0.$$

Thus, $\text{image}(\phi) \subseteq \{B \in M_2\mathbb{R} \mid \text{trace}(B) = 0\}$.

Conversely, suppose we have a matrix B with $\text{trace}(B) = 0$; we must show that $B \in \text{image}(\phi)$. We thus need to find a matrix A such that $\phi(A) = B$. In fact we have $\phi(B) = B - \frac{1}{2}\,\text{trace}(B)I = B - 0.I = B$, so we can just take $A = B$. This completes the proof that $\text{image}(\phi) = \{B \mid \text{trace}(B) = 0\}$.

Aside: If we had not noticed that $\phi(B) = B$, what would we have done? We would have $B = \left[\begin{smallmatrix} p & q \\ r & -p \end{smallmatrix}\right]$ for some p, q, r, and we would need to find a matrix $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ with $\phi(A) = B$. Using part (a) we see that this reduces to the equations $(a - d)/2 = p$ and $b = q$ and $c = r$ and $(d - a)/2 = -p$. These can be solved to give $a = 2p + d$ and $b = q$ and $c = r$ with d arbitrary. We could take $d = 0$, giving $A = \left[\begin{smallmatrix} 2p & q \\ r & 0 \end{smallmatrix}\right]$, or we could take $d = -p$ giving $A = \left[\begin{smallmatrix} p & q \\ r & -p \end{smallmatrix}\right] = B$ as before.

Solution 4. (a) $\phi(f) = \begin{bmatrix} [ax^3/3 + bx^2/2 + cx]_{-1}^{0} \\ [ax^3/3 + bx^2/2 + cx]_{-1}^{1} \\ [ax^3/3 + bx^2/2 + cx]_{0}^{1} \end{bmatrix} = \begin{bmatrix} a/3 - b/2 + c \\ 2a/3 + 2c \\ a/3 + b/2 + c \end{bmatrix}$

(b) We have $f \in \ker(\phi)$ iff $\phi(f) = 0$ iff $a/3 - b/2 + c = 0$ and $2a/3 + 2c = 0$ and $a/3 + b/2 + c = 0$. By subtracting the first and third equations we see that this implies $b = 0$, and the second equation gives $a = -3c$, so $f(x) = ax^2 + bx + c = -3cx^2 + c = c(1 - 3x^2)$. It follows that $\ker(\phi) = \{c(1 - 3x^2) \mid c \in \mathbb{R}\}$.

(c) We need

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \phi(px + q) = \begin{bmatrix} -p/2+q \\ 2q \\ p/2+q \end{bmatrix},$$

so

$$\begin{aligned} -p/2 + q &= 1 \\ 2q &= 1 \\ p/2 + q &= 0. \end{aligned}$$

These equations have the unique solution $p = -1$ and $q = 1/2$, so $g_+(x) = \frac{1}{2} - x$.

(d) We now have $g_-(x) = \frac{1}{2} + x$, and using the formula in (a) we see that $\phi(g_-) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ as required.

(e) Now put

$$W = \{[u, v, w]^T \in \mathbb{R}^3 \mid v = u + w\} = \{[u, u + w, w]^T \mid u, w \in \mathbb{R}\}.$$

The claim is that $W = \text{image}(\phi)$. Firstly, for any f we certainly have

$$\int_{-1}^{1} f(x)\, dx = \int_{-1}^{0} f(x)\, dx + \int_{0}^{1} f(x)\, dx,$$

so $\phi(f) \in W$, which proves that $\text{image}(\phi) \subseteq W$. On the other hand, given a vector $\mathbf{x} = [u, u + w, w]^T \in W$, we note that

$$\phi(ug_+ + wg_-) = u\phi(g_+) + w\phi(g_-) = u \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} u \\ u+w \\ w \end{bmatrix} = \mathbf{x},$$

so $\mathbf{x} \in \text{image}(\phi)$. This shows that $W \subseteq \text{image}(\phi)$, so $W = \text{image}(\phi)$.

Solution 5. (a) This map is injective but not surjective, and so is not an isomorphism. Indeed, if $\begin{bmatrix} x \\ y \\ x \end{bmatrix} = 0$, then clearly $\begin{bmatrix} x \\ y \end{bmatrix} = 0$. In other words, if $\phi(\mathbf{u}) = 0$, then $\mathbf{u} = 0$, which means that $\ker(\phi) = 0$, which means that ϕ is injective. On the other hand, the vector $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is clearly not of the form $\begin{bmatrix} x \\ y \\ x \end{bmatrix}$, so it does not lie in the image of ϕ, so ϕ is not surjective.

(b) This map is surjective but not injective, and so is not an isomorphism. Indeed, the vector $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ has $\phi(\mathbf{u}) = 0$, so $\mathbf{u} \in \ker(\phi)$, so $\ker(\phi) \neq 0$, so ϕ is not injective. On the other hand, for any vector $\mathbf{v} = \begin{bmatrix} p \\ q \end{bmatrix} \in \mathbb{R}^2$ we see that

$$\phi \begin{bmatrix} p+q \\ q \\ 0 \end{bmatrix} = \begin{bmatrix} (p+q)-q \\ q-0 \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix} = \mathbf{v},$$

so $\mathbf{v} \in \text{image}(\phi)$. As this works for any $\mathbf{v} \in \mathbb{R}^2$, we deduce that $\text{image}(\phi) = \mathbb{R}^2$, so ϕ is surjective.

Aside: How did we find this? We need to find a vector $\mathbf{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ with $\phi(\mathbf{u}) = \mathbf{v} = \begin{bmatrix} p \\ q \end{bmatrix}$. This reduces to the equations $x - y = p$ and $y - z = q$, giving $x = p + q + z$ and $y = q + z$ with z arbitrary. We could take $z = 0$, giving $\mathbf{u} = [p + q, q, 0]^T$ as before. Alternatively, we could take $z = -q$ giving $\mathbf{u} = [p, 0, -q]^T$.

(c) If $f(x) = ax^2 + bx + c$ then

$$\begin{aligned} f(x) &= ax^2 + bx + c & f(0) &= c \\ f'(x) &= 2ax + b & f'(0) &= b \\ f''(x) &= 2a & f''(0) &= 2a, \end{aligned}$$

so

$$\phi(ax^2 + bx + c) = [c, b, 2a]^T.$$

Now define $\psi\colon \mathbb{R}^3 \to \mathbb{R}[x]_{\leq 2}$ by $\psi([p,q,r]^T) = \frac{1}{2}rx^2 + qx + p$. We find that $\psi(\phi(f)) = f$ (for all $f \in \mathbb{R}[x]_{\leq 2}$), and $\phi(\psi([p,q,r]^T)) = [p,q,r]^T$. This means that ψ is an inverse for ϕ, so ϕ is an isomorphism (and so is both injective and surjective).

(d) This is neither injective nor surjective, and so is not an isomorphism. It is not injective because $\phi\left[\begin{smallmatrix}1\\-1\end{smallmatrix}\right] = 0$, which means that ϕ has nonzero kernel. Moreover, the matrix $I = \left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]$ does not have the form $\left[\begin{smallmatrix}0&x+y\\x+y&0\end{smallmatrix}\right]$ for any x and y, so $I \not\in \text{image}(\phi)$, so ϕ is not surjective.

(e) This is surjective but not injective, and so is not an isomorphism. It is not injective, because if we put $g(x) = x$ then

$$\int_{-1}^{1} g(x)\,dx = \int_{-1}^{1} x\,dx = \left[\tfrac{1}{2}x^2\right]_{-1}^{1} = \tfrac{1}{2}(1^2 - (-1)^2) = 0.$$

This shows that g is a nonzero element of $\ker(\phi)$, so ϕ cannot be injective. On the other hand, given any $t \in \mathbb{R}$ we can let $h(x)$ be the constant function with value $t/2$, and we find that $\phi(h) = \int_{-1}^{1} h(x)\,dx = 2.(t/2) = t$, showing that $t \in \text{image}(\phi)$. This works for any t, so $\text{image}(\phi) = \mathbb{R}$, so ϕ is surjective.

Solution 6. Consider a vector $\mathbf{u}$ in $L \cap M$. We must have $\mathbf{u} = \left[\begin{smallmatrix}s\\2s\end{smallmatrix}\right]$ for some s (because $\mathbf{u} \in L$) and $\mathbf{u} = \left[\begin{smallmatrix}2t\\t\end{smallmatrix}\right]$ for some t (because $\mathbf{u} \in M$). This means that $s = 2t$ and $t = 2s$. If we substitute the second of these equations in the first we get $s = 4s$, so $3s = 0$, so $s = 0$, so $t = 0$ and $\mathbf{u} = \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$. This shows that $L \cap M = 0$.

Now consider an arbitrary vector $\mathbf{u} = \left[\begin{smallmatrix}x\\y\end{smallmatrix}\right] \in \mathbb{R}^2$. We want to show that this lies in $L + M$, so we must find vectors $\mathbf{v} \in L$ and $\mathbf{w} \in M$ such that $\mathbf{u} = \mathbf{v} + \mathbf{w}$. In other words, we must find numbers s and t such that $\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right] = \left[\begin{smallmatrix}s\\2s\end{smallmatrix}\right] + \left[\begin{smallmatrix}2t\\t\end{smallmatrix}\right]$, so

$$\begin{aligned} x &= s + 2t \\ y &= 2s + t. \end{aligned}$$

These equations have the (unique) solution

$$\begin{aligned} t &= (2x - y)/3 \\ s &= (2y - x)/3 \end{aligned}$$

so we can put

$$\mathbf{v} = \left[\begin{smallmatrix}s\\2s\end{smallmatrix}\right] = \left[\begin{smallmatrix}(2y-x)/3\\(4y-2x)/3\end{smallmatrix}\right] \qquad \mathbf{w} = \left[\begin{smallmatrix}2t\\t\end{smallmatrix}\right] = \left[\begin{smallmatrix}(4x-2y)/3\\(2x-y)/3\end{smallmatrix}\right].$$

We then have $\mathbf{v} \in L$ and $\mathbf{w} \in M$ and $\mathbf{u} = \mathbf{v} + \mathbf{w}$, so $\mathbf{u} \in L + M$. This works for any vector $\mathbf{u} \in \mathbb{R}^2$, so $\mathbb{R}^2 = L + M$ as claimed.

Solution 7. If $A \in V \cap W$ then $A = A^T$ (because $A \in V$) and also $A^T = -A$ (because $A \in W$) so $A = -A$. We now add A to both sides to get $2A = 0$, and divide by 2 to get $A = 0$. This shows that $V \cap W = 0$. Now consider an arbitrary matrix $A \in U$. Put $A_+ = (A + A^T)/2$ and $A_- = (A - A^T)/2$. Then $A_+ + A_- = A$. Moreover, we have

$$\begin{aligned} A_+^T &= (A^T + A^{TT})/2 = (A^T + A)/2 = (A + A^T)/2 = A_+ \\ A_-^T &= (A^T - A^{TT})/2 = (A^T - A)/2 = -(A - A^T)/2 = -A_- \end{aligned}$$

which shows that $A_+ \in V$ and $A_- \in W$. As $A = A_+ + A_-$ with $A_+ \in V$ and $A_- \in W$, we have $A \in V + W$. This works for any $A \in U$, so $U = V + W$ as claimed.