Vector Spaces and Fourier Theory — Problem Sheet 4

Solution 1. (a) These vectors are linearly independent. Indeed, we have

$$\lambda_1\mathbf{u}_1 + \lambda_2\mathbf{u}_2 + \lambda_3\mathbf{u}_3 = [\lambda_1, 2\lambda_2, 3\lambda_3, 2\lambda_2, \lambda_1]^T,$$

and this can only be zero if $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Thus, the only linear relation between $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$ is the trivial one, as required.

(b) These are linearly dependent, because of the nontrivial relation $4\mathbf{v}_1 - 2\mathbf{v}_2 - \mathbf{v}_3 = 0$.

(c) These are linearly independent. Indeed, suppose we have a relation $\lambda_1\mathbf{w}_1+\lambda_2\mathbf{w}_2+\lambda_3\mathbf{w}_3 = 0$. This means that

$$\lambda_1\begin{bmatrix}1\\1\\2\end{bmatrix} + \lambda_2\begin{bmatrix}4\\5\\7\end{bmatrix} + \lambda_3\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix},$$

so

$$\begin{aligned}\lambda_1 + 4\lambda_2 + \lambda_3 &= 0\\ \lambda_1 + 5\lambda_2 + \lambda_3 &= 0\\ 2\lambda_1 + 7\lambda_2 + \lambda_3 &= 0.\end{aligned}$$

Subtracting the first two equations gives $\lambda_2 = 0$. Given this, we can subtract the last two equations to get $\lambda_1 = 0$. Feeding this back into the first equation gives $\lambda_3 = 0$. Thus, the only linear relation between $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$ is the trivial one, as required.

This can also be done by matrix methods. Let A be the matrix whose columns are $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$, so

$$A = \begin{bmatrix}1&4&1\\1&5&1\\2&7&1\end{bmatrix}.$$

Then $\det(A) = -1 \neq 0$, and if we row-reduce eihter A or A^T then we get the identity. Any of these facts implies that $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$ are linearly independent, as you should remember from SOM201.

Solution 2. (a)

$$\begin{aligned}\mu_{\mathcal{V}}([0,1,1,-1]^T) &= 0.1 + 1.x + 1.(1+x)^2 - 1.(1+x^2)\\ &= x + 1 + 2x + x^2 - 1 - x^2 = 3x.\end{aligned}$$

(b) Here it is simplest to just observe that

$$x^2 = -1 + (1+x^2) = (-1).1 + 0.x + 0.(1+x)^2 + 1.(1+x^2) = \mu_{\mathcal{V}}([-1,0,0,1]^T).$$

For a more laborious but systematic approach, we have

$$\begin{aligned}\mu_{\mathcal{V}}(\boldsymbol{\lambda}) &= \lambda_1.1 + \lambda_2.x + \lambda_3.(1+2x+x^2) + \lambda_4.(1+x^2)\\ &= (\lambda_1+\lambda_3+\lambda_4) + (\lambda_2+2\lambda_3)x + (\lambda_3+\lambda_4)x^2.\end{aligned}$$

We want this to equal x^2, so we must have

$$\begin{aligned}\lambda_1 + \lambda_3 + \lambda_4 &= 0\\ \lambda_2 + 2\lambda_3 &= 0\\ \lambda_3 + \lambda_4 &= 1.\end{aligned}$$

These equations can be solved to give $\lambda_1 = -1$ and $\lambda_2 = -2\lambda_3$ and $\lambda_4 = 1 - \lambda_3$ (where λ_3 can be anything). It is simplest to take $\lambda_3 = 0$, so $\lambda_1 = -1$ and $\lambda_2 = 0$ and $\lambda_4 = 1$, so $\boldsymbol{\lambda} = ([-1,0,0,1]^T)$.

(c) Again it is easiest to just observe that $(1+x)^2 = (1+x^2)+2x$, so $0.1+2.x-1.(1+x)^2+1.(1+x^2) = 0$, so $\mu_{\mathcal{V}}([0,2,-1,1]^T) = 0$. For a more laborious but systematic approach, recall that

$$\mu_{\mathcal{V}}(\boldsymbol{\lambda}) = (\lambda_1+\lambda_3+\lambda_4) + (\lambda_2+2\lambda_3)x + (\lambda_3+\lambda_4)x^2.$$

We want this to equal 0, so we must have

$$\begin{aligned} \lambda_1 + \lambda_3 + \lambda_4 &= 0 \\ \lambda_2 + 2\lambda_3 &= 0 \\ \lambda_3 + \lambda_4 &= 0. \end{aligned}$$

These equations can be solved to give $\lambda_1 = 0$ and $\lambda_3 = -\lambda_4$ and $\lambda_2 = -2\lambda_3 = 2\lambda_4$, so $\boldsymbol{\lambda} = \lambda_4.[0, 2, -1, 1]^T$. Here λ_4 can be anything, but it is simplest to take $\lambda_4 = 1$ to get $\boldsymbol{\lambda} = [0, 2, -1, 1]^T$.

Solution 3. (a) As the matrices in $\mathcal{A}$ all have 0 in the top left corner, the same will be true of any matrix in span($\mathcal{A}$). (The formula is

$$\mu_{\mathcal{A}}(\boldsymbol{\lambda}) = \lambda_1 \left[\begin{smallmatrix} 0 & 1 \\ 2 & 3 \end{smallmatrix}\right] + \lambda_2 \left[\begin{smallmatrix} 0 & 2 \\ 1 & 3 \end{smallmatrix}\right] + \lambda_3 \left[\begin{smallmatrix} 0 & 1 \\ 3 & 2 \end{smallmatrix}\right] + \lambda_4 \left[\begin{smallmatrix} 0 & 3 \\ 2 & 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & \lambda_1+2\lambda_2+\lambda_3+3\lambda_4 \\ 2\lambda_1+\lambda_2+3\lambda_3+2\lambda_4 & 3\lambda_1+3\lambda_2+2\lambda_3+\lambda_4 \end{smallmatrix}\right],$$

but you should be able to follow the argument without needing the formula.) In particular, the identity matrix cannot lie in span($\mathcal{A}$), because it does not have 0 in the top left corner. Thus span($\mathcal{A}$) $\neq M_2\mathbb{R}$.

(b) As the matrices in $\mathcal{B}$ are symmetric, the same will be true of any matrix in span($\mathcal{B}$). (The formula is

$$\mu_{\mathcal{B}}(\boldsymbol{\lambda}) = \lambda_1 \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right] + \lambda_2 \left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right] + \lambda_3 \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right] + \lambda_4 \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} \lambda_1+\lambda_3+\lambda_4 & \lambda_1+\lambda_2 \\ \lambda_1+\lambda_2 & \lambda_2+\lambda_3-\lambda_4 \end{smallmatrix}\right],$$

but you should be able to follow the argument without needing the formula.) In particular, the matrix $\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ cannot lie in span($\mathcal{B}$), because it is not symmetric. Thus span($\mathcal{B}$) $\neq M_2\mathbb{R}$.

(c) The list $\mathcal{C}$ spans $M_2\mathbb{R}$. To see this, consider a matrix $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. We have

$$\mu_{\mathcal{C}}(\boldsymbol{\lambda}) = \lambda_1 \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right] + \lambda_2 \left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right] + \lambda_3 \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right] + \lambda_4 \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} \lambda_1+\lambda_2+\lambda_3+\lambda_4 & \lambda_2+\lambda_3+\lambda_4 \\ \lambda_3+\lambda_4 & \lambda_4 \end{smallmatrix}\right].$$

We want this to equal $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, so we must have

$$\begin{aligned} \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 &= a \\ \lambda_2 + \lambda_3 + \lambda_4 &= b \\ \lambda_3 + \lambda_4 &= c \\ \lambda_4 &= d \end{aligned}$$

These equations have the (unique) solution $\lambda_1 = a - b$, $\lambda_2 = b - c$, $\lambda_3 = c - d$ and $\lambda_4 = d$. In conclusion, we have

$$\mu_{\mathcal{C}}([a-b, b-c, c-d, d]^T) = A,$$

showing that $A \in$ span($\mathcal{C}$). This works for any matrix A, so $M_2\mathbb{R} =$ span($\mathcal{C}$).

(d) As all the matrices in $\mathcal{D}$ have trace zero, the same will be true of any matrix in span($\mathcal{D}$). In particular, the identity matrix cannot lie in span($\mathcal{D}$), because it does not have trace zero. Thus, span($\mathcal{D}$) $\neq M_2\mathbb{R}$.

Solution 4. Given $\boldsymbol{\lambda} = [\lambda_0, \lambda_1, \lambda_2]^T \in \mathbb{R}^3$, we have

$$\begin{aligned} \mu_{\mathcal{R}}(\boldsymbol{\lambda})(x) &= \lambda_0 x^2 + \lambda_1 (x+1)^2 + \lambda_2 (x+2)^2 = \lambda_0 x^2 + \lambda_1 (x^2 + 2x + 1) + \lambda_2 (x^2 + 4x + 4) \\ &= (\lambda_0 + \lambda_1 + \lambda_2) x^2 + (2\lambda_1 + 4\lambda_2) x + (\lambda_1 + 4\lambda_2) \end{aligned}$$

Suppose we have a quadratic polynomial $q(x) = ax^2 + bx + c$, and we want to have $\mu_{\mathcal{R}}(\boldsymbol{\lambda}) = q$. We must then have

$$\begin{aligned} \lambda_0 + \lambda_1 + \lambda_2 &= a \\ 2\lambda_1 + 4\lambda_2 &= b \\ \lambda_1 + 4\lambda_2 &= c. \end{aligned}$$

Subtracting the last two equations gives $\lambda_1 = b - c$, and we can put this into the last equation to give $\lambda_2 = c/2 - b/4$. We then put these two values back into the first equation to give $\lambda_0 = a - 3b/4 + c/2$. The conclusion is that

$$\mu\left(\begin{bmatrix} a-3b/4+c/2 \\ b-c \\ c/2-b/4 \end{bmatrix}\right) = q,$$

showing that $q \in \text{span}(\mathcal{R})$. As this works for any quadratic polynomial q, we have $\text{span}(\mathcal{R}) = \mathbb{R}[x]_{\leq 2}$.

Solution 5. Suppose we have a linear relation $\lambda f + \mu g + \nu h = 0$. Note that the symbol 0 on the right hand side means the zero function, which takes the value 0 for all x. In particular, it takes the value 0 at $x = a$, so we have

$$\lambda f(a) + \mu g(a) + \nu h(a) = 0.$$

As $f(a) = 1$ and $g(a) = h(a) = 0$, this simplifies to $\lambda = 0$. Similarly, we have

$$\lambda f(b) + \mu g(b) + \nu h(b) = 0$$
$$\lambda f(c) + \mu g(c) + \nu h(c) = 0,$$

and these simplify to give $\mu = \nu = 0$. Thus, the only linear relation between f, g and h is the trivial one, so they are linearly independent.

Solution 6. We have $f_k'(x) = ke^{kx}$ and $f_k''(x) = k^2 e^{kx}$, so

$$\begin{aligned} W(f_1, f_2, f_3)(x) &= \det \begin{bmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{bmatrix} = e^x e^{2x} e^{3x} \det \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \\ &= e^{6x}\left(\det \begin{bmatrix} 2 & 3 \\ 4 & 9 \end{bmatrix} - \det \begin{bmatrix} 1 & 3 \\ 1 & 9 \end{bmatrix} + \det \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix}\right) = e^{6x}(6 - 6 + 2) = 2e^{6x}. \end{aligned}$$

This is not the zero function, so f_1, f_2 and f_3 are linearly independent.

Solution 7. The Wronskian matrix is

$$WM = \begin{bmatrix} x^n & x^{n+1} & x^{n+2} \\ nx^{n-1} & (n+1)x^n & (n+2)x^{n+1} \\ n(n-1)x^{n-2} & n(n+1)x^{n-1} & (n+1)(n+2)x^n \end{bmatrix}$$

We can extract a factor of x^n from the first row, x^{n-1} from the second row, and x^{n-2} from the third row, to get

$$W = \det(WM) = x^{3n-3} \det \begin{bmatrix} 1 & x & x^2 \\ n & (n+1)x & (n+2)x^2 \\ n(n-1) & n(n+1)x & (n+1)(n+2)x^2 \end{bmatrix}.$$

We then extract x from the second column, and x^2 from the third column, to get $W = x^{3n}\det(V)$, where

$$V = \begin{bmatrix} 1 & 1 & 1 \\ n & n+1 & n+2 \\ n(n-1) & n(n+1) & (n+1)(n+2) \end{bmatrix}.$$

We will expand $\det(V)$ along the top row, using the cofactors

$$\det\begin{bmatrix} (n+1) & (n+2) \\ n(n+1) & (n+1)(n+2) \end{bmatrix} = (n+1)^2(n+2) - n(n+1)(n+2) = (n+1)(n+2) = n^2 + 3n + 2$$
$$\det\begin{bmatrix} n & (n+2) \\ n(n-1) & (n+1)(n+2) \end{bmatrix} = n(n+1)(n+2) - n(n-1)(n+2) = 2n(n+2) = 2n^2 + 4n$$
$$\det\begin{bmatrix} n & (n+1) \\ n(n-1) & n(n+1) \end{bmatrix} = n^2(n+1) - n(n-1)(n+1) = n(n+1) = (n^2 + n).$$

Thus

$$\det(V) = 1.(n^2 + 3n + 2) - 1.(2n^2 + 4n) + 1.(n^2 + n) = n^2 + 3n + 2 - 2n^2 - 4n + n^2 + n = 2,$$

so $W = 2x^{3n}$. Alternatively, you could ask Maple:

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with(LinearAlgebra):

WM := simplify(
      <<      x^n     ,       x^(n+1)      ,       x^(n+2)      >|
       < diff(x^n,x)  , diff(x^(n+1),x)   , diff(x^(n+2),x)    >|
       < diff(x^n,x,x), diff(x^(n+1),x,x), diff(x^(n+2),x,x) >>
      );

W := simplify(Determinant(WM));
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Solution 8. Consider a matrix

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}$$

Explicitly, we have

$$\begin{aligned} \phi(A) &= \begin{bmatrix} 1 & x & x^2 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & x & x^2 \end{bmatrix} \begin{bmatrix} a_1 + a_2 x + a_3 x^2 \\ a_4 + a_5 x + a_6 x^2 \\ a_7 + a_8 x + a_9 x^2 \end{bmatrix} \\ &= a_1 + (a_2 + a_4)x + (a_3 + a_5 + a_7)x^2 + (a_6 + a_8)x^3 + a_9 x^4 \end{aligned}$$

In particular, given any element $f(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4$ in $\mathbb{R}[x]_{\leq 4}$, we can take

$$a_1 = b_0,\ a_2 = b_1,\ a_3 = b_2,\ a_6 = b_3,\ a_9 = b_4,\ a_4 = a_5 = a_7 = a_8 = 0$$

and we then have $\phi(A) = f$. More explicitly:

$$\phi \begin{bmatrix} b_0 & b_1 & b_2 \\ 0 & 0 & b_3 \\ 0 & 0 & b_4 \end{bmatrix} = \begin{bmatrix} 1 & x & x^2 \end{bmatrix} \begin{bmatrix} b_0 & b_1 & b_2 \\ 0 & 0 & b_3 \\ 0 & 0 & b_4 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix} = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4.$$

This means that ϕ is surjective. The kernel is the set of matrices A for which

$$a_1 = a_2 + a_4 = a_3 + a_5 + a_7 = a_6 + a_8 = a_9 = 0,$$

or in other words, the set of matrices of the form

$$A = \begin{bmatrix} 0 & a_2 & a_3 \\ -a_2 & a_5 & a_6 \\ -a_3 - a_5 & -a_6 & 0 \end{bmatrix} = a_2 \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_3 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + a_5 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} + a_6 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

It follows that the following matrices form a basis for $\ker(\phi)$:

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

Solution 9. (a) Given any vector $\mathbf{v} = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \in \mathbb{R}^2$, we can define $f(x) = a + (b - a)x$; then $f \in \mathbb{R}[x]_{\leq 3}$ and $f(0) = a$ and $f(1) = b$, so $\phi(f) = \mathbf{v}$. This shows that ϕ is surjective.

Now consider a polynomial $f(x) = ax^3 + bx^2 + cx + d$. We have $\phi(f) = 0$ iff $f(1) = 0 = f(0)$ iff $a + b + c + d = 0 = d$ iff $c = -a - b$ and $d = 0$. If this holds then

$$f(x) = ax^3 + bx^2 + (-a - b)x = a(x^3 - x) + b(x^2 - x) = a(x^3 - x^2) + (a + b)(x^2 - x).$$

In other words, if we put $p(x) = x^3 - x^2$ and $q(x) = x^2 - x$, then $\phi(p) = \phi(q) = 0$ and $f = a\,p + (a + b)\,q \in \operatorname{span}(p, q)$. This shows that p and q span $\ker(\phi)$, and they are clearly linearly independent, so they give a basis for $\ker(\phi)$.

(b) If $\psi(f) = 0$ then we have $f(0) = f(1) = f(2) = f(3) = 0$, so $f(x)$ has at least four different roots. As f is a polynomial of degree at most three, this is impossible, unless $f = 0$. To be more explicit, suppose that $f(x) = ax^3 + bx^2 + cx + d$ and $f(0) = f(1) = f(2) = f(3) = 0$. This means that

$$\begin{aligned} d &= 0 \\ a + b + c + d &= 0 \\ 8a + 4b + 2c + d &= 0 \\ 27a + 9b + 3c + d &= 0, \end{aligned}$$

and these equations can be solved in the standard way to show that $a = b = c = d = 0$.

Solution 10. First consider the matrices

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

We have

$$\begin{aligned} \phi(E_1) &= \begin{bmatrix} p & q \end{bmatrix} \begin{bmatrix} r \\ 0 \end{bmatrix} = pr \\ \phi(E_2) &= \begin{bmatrix} p & q \end{bmatrix} \begin{bmatrix} s \\ 0 \end{bmatrix} = ps \\ \phi(E_3) &= \begin{bmatrix} p & q \end{bmatrix} \begin{bmatrix} 0 \\ r \end{bmatrix} = qr \\ \phi(E_4) &= \begin{bmatrix} p & q \end{bmatrix} \begin{bmatrix} 0 \\ s \end{bmatrix} = qs. \end{aligned}$$

On the other hand, we have $\mathrm{trace}(E_1) = \mathrm{trace}(E_4) = 1$ and $\mathrm{trace}(E_2) = \mathrm{trace}(E_3) = 0$. Suppose for a contradiction that we have $\phi(A) = \mathrm{trace}(A)$. By taking $A = E_i$ for $i = 1, 2, 3, 4$ we get $pr = 1$ and $ps = 0$ and $qr = 0$ and $qs = 1$. As $pr = qs = 1$ we see that all of p, q, r and s must be nonzero. This conflicts with the equations $ps = 0 = qr$, so we have the required contradiction.