Vector Spaces and Fourier Theory — Problem Sheet 5

Solution 1. We have

$$\phi(\mathbf{e}_1) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \qquad \phi(\mathbf{e}_2) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad \phi(\mathbf{e}_3) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

so the matrix with respect to the standard basis is

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

On the other hand, we have

$$\begin{aligned}
\phi(\mathbf{u}_1) &= \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = 2.\mathbf{u}_1 + 0.\mathbf{u}_2 + 0.\mathbf{u}_3 \\
\phi(\mathbf{u}_2) &= \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = 0\mathbf{u}_1 - 1.\mathbf{u}_2 + 0.\mathbf{u}_3 \\
\phi(\mathbf{u}_3) &= \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = 0\mathbf{u}_1 + 0.\mathbf{u}_2 - 1.\mathbf{u}_3
\end{aligned}$$

so the matrix with respect to $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$ is

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Solution 2. The usual basis of $\mathbb{R}[x]_{\leq 2}$ is $\{1, x, x^2\}$. If $f(x) = x^2$ then $f'(x) = 2x$ and $f''(x) = 2$, so $f(0) = 0$ and $f'(1) = 2$ and $f''(2) = 2$, so $\phi(f) = [0, 2, 2]^T$. In the same way, we get

$$\phi(1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \phi(x) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \phi(x^2) = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}.$$

The matrix of ϕ has these three vectors as its columns, so the matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$.

Solution 3.

(a) The functions $f_0(x) = e^{\lambda x}$, $f_1(x) = xe^{\lambda x}$ and $f_2(x) = x^2 e^{\lambda x}$ form a basis for V.

(b) If $f \in V$ then $f(x) = (ax^2 + bx + c)e^{\lambda x}$, so

$$f'(x) = (2ax + b)e^{\lambda x} + (ax^2 + bx + c)\lambda e^{\lambda x} = (a\lambda x^2 + (2a + b\lambda)x + (b + c\lambda))e^{\lambda x}.$$

Here a, b, c and λ are all just constants, so we see that $f'(x)$ is again a quadratic polynomial times $e^{\lambda x}$, so $f' \in V$ as required.

(c) We have

$$\begin{aligned}
f_0' &= \lambda f_0 + 0.f_1 + 0.f_2 \\
f_1' &= 1.f_0 + \lambda f_1 + 0.f_2 \\
f_2' &= 0.f_0 + 2.f_1 + \lambda f_2
\end{aligned}$$

so the matrix is $\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 2 \\ 0 & 0 & \lambda \end{bmatrix}$.

(d) We have $(D - \lambda)f_0 = f_0' - \lambda f_0 = 0$ and similarly $(D - \lambda)f_1 = f_0$ and $(D - \lambda)f_2 = 2f_1$. It follows that $(D - \lambda)^2 f_2 = (D - \lambda)f_1 = 0$ and $(D - \lambda)^3 f_2 = 2(D - \lambda)^2 f_1 = 0$, so $(D - \lambda)^3 f_i = 0$ for $i = 0, 1, 2$, so $(D - \lambda)^3 = 0$. Alternatively, we can note that $D - \lambda$ has matrix $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$, and it is easy to see that the cube of this matrix is zero.

Solution 4. We have $J^2 = -I$, so if $f(x) = ax^2 + bx + c$ we have $\phi(f) = (c - a)I + bJ = \begin{bmatrix} c - a & b \\ -b & c - a \end{bmatrix}$. In particular, we have $\phi(f) = 0$ iff $b = 0$ and $c = a$, which means that $f(x) = a(x^2 + 1)$. We also see that image(ϕ) is spanned by I and J, which are linearly independent. Thus $\{x^2 + 1\}$ is a basis for $\ker(\phi)$ and $\{I, J\}$ is a basis for image(ϕ).

Solution 5. (a) If $v_1, \dots, v_n$ are linearly dependent, then there must exist a linear relation $\lambda_1 v_1 + \cdots + \lambda_n v_n = 0$ in which one of the λ_i's is nonzero. We then have

$$\lambda_1 \phi(v_1) + \cdots + \lambda_n \phi(v_n) = \phi(\lambda_1 v_1 + \cdots + \lambda_n v_n) = \phi(0) = 0,$$

which gives a nontrivial linear relation between the elements $\phi(v_1), \dots, \phi(v_n)$, showing that they too are linearly dependent.

(b) Consider $\phi\colon \mathbb{R}^2 \to \mathbb{R}$ given by $\phi\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = x + y$, and the vectors $v_1 = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ and $v_2 = \left[\begin{smallmatrix} 0 \\ -1 \end{smallmatrix}\right]$. Then v_1 and v_2 are linearly independent, but $\phi(v_1) + \phi(v_2) = 0$, which shows that $\phi(v_1)$ and $\phi(v_2)$ are linearly dependent.

Of course there are many other examples. The minimal example is to let ϕ be the map $\mathbb{R} \to 0$ given by $\phi(t) = 0$ for all t, and $n = 1$, and $v_1 = 1 \in \mathbb{R}$. But this is perhaps so simple as to be confusing.

(c) This is logically equivalent to (a). If $\phi(v_1), \dots, \phi(v_n)$ are linearly independent, then $v_1, \dots, v_n$ cannot be dependent (as that would contradict (a)), so they must be linearly independent.

Solution 6. Put $u = \phi(x) \in V$ and $v = \phi(1) \in V$. Then

$$\phi(ax + b) = \phi(a.x + b.1) = a\phi(x) + b\phi(1) = au + bv,$$

as required.

Solution 7. Firstly, V is just the set of odd polynomials of degree at most three. Any such polynomial has the form $ax^3 + cx$ for some $a, c \in \mathbb{R}$, so $\{x^3, x\}$ is a basis for V. Next, consider a polynomial $f(x) = ax^3 + bx^2 + cx + d$. We then have $f''(1) = 6a + 2b$ and $f'(1) = 3a + 2b + c$ and $f(1) = a + b + c + d$, so $f \in W$ iff $6a + 2b = 6a + 4b + 2c = 6a + 6b + 6c + 6d$. Note that $6a + 2b = 6a + 4b + 2c$ iff $2b + 2c = 0$, and $6a + 4b + 2c = 6a + 6b + 6c + 6d$ iff $2b + 4c + 6d = 0$. This means that $f \in W$ iff $2b + 2c = 0 = 2b + 4c + 6d$, or equivalently $c = -b$ and $d = b/3$. It follows that W is the set of polynomials of the form

$$f(x) = ax^3 + bx^2 - bx + b/3 = ax^3 + b(x^2 - x + 1/3).$$

This means that $\{x^3, x^2 - x + 1/3\}$ is a basis for W. Next note that a polynomial of the above form can only be odd if $b = 0$, so $f(x) = ax^3$. This means that $V \cap W$ is the set of polynomials of the form ax^3, so $\{x^3\}$ is a basis for $V \cap W$.

Now put $U = \{f \in \mathbb{R}[x]_{\leq 3} \mid f''(0) - 6f(0) = 0\}$. If $f(x) = ax^3 + bx^2 + cx + d$ then $f''(0) - 6f(0) = 2b - 6d$, so $f \in U$ iff $d = b/3$. Using this criterion we see that x^3, x and $x^2 - x + 1/3$ lie in U. As these three polynomials contain a basis for V and also a basis for W, we see that $V + W \leq U$. Conversely, suppose we have $f(x)$ as above in U, so $d = b/3$. This means that

$$f(x) = ax^3 + b(x^2 - x + 1/3) + (b + c)x \in \operatorname{span}\{x^3, x^2 - x + 1/3, x\} = V + W.$$

This shows that $U \leq V + W$, so in fact $U = V + W$ as claimed.

Solution 8. We have $\Delta(x^k) = (x+1)^k - x^k$, so

$$\begin{aligned}
\Delta(x^0) &= 0 && = 0.x^0 + 0.x^1 + 0.x^2 + 0.x^3 \\
\Delta(x^1) &= 1 && = 1.x^0 + 0.x^1 + 0.x^2 + 0.x^3 \\
\Delta(x^2) &= 2x + 1 && = 1.x^0 + 2.x^1 + 0.x^2 + 0.x^3 \\
\Delta(x^3) &= 3x^2 + 3x + 1 && = 1.x^0 + 3.x^1 + 3.x^2 + 0.x^3 \\
\Delta(x^4) &= 4x^3 + 6x^2 + 4x + 1 && = 1.x^0 + 4.x^1 + 6.x^2 + 4.x^3
\end{aligned}$$

The matrix is therefore

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

It is clear that the nonzero columns form a basis for $\mathbb{R}^4$. It follows that the map is surjective (so the image is all of $\mathbb{R}[x]_{\leq 3}$), and the kernel is just the set of constant polynomials. More explicitly, consider a polynomial $p = ax^4 + bx^3 + cx^2 + dx + e$. We have

$$\begin{aligned}\Delta(p) &= a(4x^3 + 6x^2 + 4x + 1) + b(3x^2 + 3x + 1) + c(2x + 1) + d \\ &= 4ax^3 + (6a + 3b)x^2 + (4a + 3b + 2c)x + (a + b + c + d).\end{aligned}$$

We can only have $\Delta(p) = 0$ if $4a = 6a + 3b = 4a + 3b + 2c = a + b + c + d = 0$, which is easily solved to give $a = b = c = d = 0$ (with e arbitrary). In other words, $\Delta(p)$ can only be zero if p is constant. We also find that

$$\begin{aligned}\Delta(x) &= 1 \\ \Delta((x^2 - x)/2) &= x \\ \Delta((2x^3 - 3x^2 + x)/6) &= x^2 \\ \Delta((x^4 - 2x^3 + x^2)/4) &= x^3,\end{aligned}$$

so the image of Δ is a subspace containing the elements $1, x, x^2$ and x^3, but these elements span all of $\mathbb{R}[x]_{\leq 3}$, so Δ is surjective.