# Vector Spaces and Fourier Theory — Problem Sheet 6

**Solution 1.** (a) The general formula is

$$\alpha\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4z \\ -3z \\ 3y-4x \end{bmatrix},$$

so

$$\begin{aligned}
\alpha(\mathbf{u}_1) &= \begin{bmatrix} 60 \\ -45 \\ -100 \end{bmatrix} = 5\mathbf{u}_3 && = 0.\mathbf{u}_1 + 0.\mathbf{u}_2 + 5.\mathbf{u}_3 \\
\alpha(\mathbf{u}_2) &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} && = 0.\mathbf{u}_1 + 0.\mathbf{u}_2 + 0.\mathbf{u}_3 \\
\alpha(\mathbf{u}_3) &= \begin{bmatrix} -80 \\ 60 \\ -75 \end{bmatrix} = -5\mathbf{u}_1 && = -5\mathbf{u}_1 + 0.\mathbf{u}_2 + 0.\mathbf{u}_3
\end{aligned}$$

(b) The lists of coefficients here form the columns of the matrix $A$, so $A = \begin{bmatrix} 0 & 0 & -5 \\ 0 & 0 & 0 \\ 5 & 0 & 0 \end{bmatrix}$.

(c)

$$\begin{aligned}
\mu_{\mathcal{U}}(\mathbf{b}) &= 1.\mathbf{u}_1 + 2.\mathbf{u}_2 + 3.\mathbf{u}_3 = \begin{bmatrix} 16 \\ -12 \\ 15 \end{bmatrix} + \begin{bmatrix} 30 \\ 40 \\ 0 \end{bmatrix} + \begin{bmatrix} 36 \\ -27 \\ -60 \end{bmatrix} = \begin{bmatrix} 82 \\ 1 \\ -45 \end{bmatrix} \\
\alpha(\mu_{\mathcal{U}}(\mathbf{b})) &= \alpha\begin{bmatrix} 82 \\ 1 \\ -45 \end{bmatrix} = \begin{bmatrix} 4\times(-45) \\ (-3)\times(-45) \\ 3\times1-4\times82 \end{bmatrix} = \begin{bmatrix} -180 \\ 135 \\ -325 \end{bmatrix} \\
\phi_A(\mathbf{b}) &= A\mathbf{b} = \begin{bmatrix} 0 & 0 & -5 \\ 0 & 0 & 0 \\ 5 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -15 \\ 0 \\ 5 \end{bmatrix} \\
\mu_{\mathcal{U}}(\phi_A(\mathbf{b})) &= \mu_{\mathcal{U}}\begin{bmatrix} -15 \\ 0 \\ 5 \end{bmatrix} = -15.\mathbf{u}_1 + 0.\mathbf{u}_2 + 5.\mathbf{u}_3 = \begin{bmatrix} (-15)\times16+5\times12 \\ (-15)\times(-12)+5\times(-9) \\ (-15)\times15+5\times(-20) \end{bmatrix} = \begin{bmatrix} -180 \\ 135 \\ -325 \end{bmatrix}
\end{aligned}$$

**Solution 2.** (a) We have $E_1^T = E_1$ and $E_2^T = E_3$ and $E_3^T = E_2$ and $E_4^T = E_4$. It follows that $\alpha(E_1) = \alpha(E_4) = 0$, whereas $\alpha(E_2) = E_2 - E_3$ and $\alpha(E_3) = E_3 - E_2$. From this it follows that

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(b)

$$\begin{aligned}
\beta(E_1) &= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = 1.E_1 + 0.E_2 + 1.E_3 + 0.E_4 \\
\beta(E_2) &= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = 0.E_1 + 1.E_2 + 0.E_3 + 1.E_4 \\
\beta(E_3) &= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = 1.E_1 + 0.E_2 + 1.E_3 + 0.E_4 \\
\beta(E_4) &= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = 0.E_1 + 1.E_2 + 0.E_3 + 1.E_4
\end{aligned}$$

The lists of coefficents here give the columns of $B$, so

$$B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

(c) Using part (b) we get

$$\begin{aligned}
\alpha\beta(E_1) &= \alpha\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = 0.E_1 - E_2 + E_3 + 0.E_4 \\
\alpha\beta(E_2) &= \alpha\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = 0.E_1 + E_2 - E_3 + 0.E_4 \\
\alpha\beta(E_3) &= \alpha\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = 0.E_1 - E_2 + E_3 + 0.E_4 \\
\alpha\beta(E_4) &= \alpha\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = 0.E_1 + E_2 - E_3 + 0.E_4
\end{aligned}$$

The lists of coefficents here give the columns of $C$, so

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(d) One checks directly that

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

so $AB = C$.

**Solution 3.** (a) The columns of $P$ are the lists of coefficents in the following equations:

$$\begin{aligned} E'_1 &= E_1 + 0.E_2 + 0.E_3 + E_4 \\ E'_2 &= E_1 + 0.E_2 + 0.E_3 - E_4 \\ E'_3 &= 0.E_1 + E_2 + E_3 + 0.E_4 \\ E'_4 &= 0.E_1 + E_2 - E_3 + 0.E_4 \end{aligned}$$

Thus,

$$P = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

(b) For $i \leq 3$, the matrix $E'_i$ is symmetric, so $\alpha(E'_i) = 0$. This means that the first three columns of $A'$ are zero. We also have

$$\alpha(E'_4) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} = 2E'_4 = 0.E'_1 + 0.E'_2 + 0.E'_3 + 2.E'_4,$$

so the last column of $A'$ is $[0,0,0,2]^T$, so

$$A' = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

(c) Just by multiplying out we see that

$$PA' = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 \end{bmatrix} = AP$$

**Solution 4.** The obvious basis to use is $x^2, x, 1$. We have

$$\begin{aligned} \phi(x^2) &= x^2 + 2x + 2 \\ \phi(x) &= 0.x^2 + x + 1 \\ \phi(1) &= 0.x^2 + 0.x + 1, \end{aligned}$$

so the matrix of $\phi$ with respect to $x^2, x, 1$ is

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$$

We thus have

$$\begin{aligned} \mathrm{trace}(\phi) &= \mathrm{trace}(P) = 1 + 1 + 1 = 3 \\ \det(\phi) &= \det(P) = 1.\det\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} - 0.\det\begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix} + 0.\det\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} = 1 \\ \mathrm{char}(\phi)(t) &= \mathrm{char}(P)(t) = \det\begin{bmatrix} t-1 & 0 & 0 \\ -2 & t-1 & 0 \\ -2 & -1 & t-1 \end{bmatrix} = (t-1)^3. \end{aligned}$$

**Solution 5.** (1) Clearly $\mathbf{v}_1 \not\in 0 = V_0$, so 1 is a jump.

(2) Clearly $\mathbf{v}_2$ is not a multiple of $\mathbf{v}_1$, so $\mathbf{v}_2 \not\in V_1$, so 2 is a jump.

(3) We have $\mathbf{v}_3 = (\mathbf{v}_1 - \mathbf{v}_2)/2 \in V_2$, so 3 is not a jump.

(4) We have $\mathbf{v}_4 = \mathbf{v}_1 - \mathbf{v}_3 \in V_3$, so 4 is not a jump.

(5) The vectors $\mathbf{v}_1, \dots, \mathbf{v}_4$ all have the property that the second and third coordinates are the same. Any vector in $V_4 = \mathrm{span}(\mathbf{v}_1, \dots, \mathbf{v}_4)$ will therefore have the same property. However, the second and third coordinates in $\mathbf{v}_5$ are different, so $\mathbf{v}_5 \not\in V_4$, so 5 is a jump.

(6) We have $\mathbf{v}_6 = \mathbf{v}_1 - \mathbf{v}_5 \in V_5$, so 6 is not a jump.

(7) The vector $\mathbf{v}_7$ does not lie in $V_6$. The cleanest way prove this is to consider the linear map $\phi\colon \mathbb{R}^6 \to \mathbb{R}$ give by

$$\phi([x_1, x_2, x_3, x_4, x_5, x_6]^T) = x_2 - x_3 + x_4 - x_5.$$

We then find that $\phi(\mathbf{v}_1) = \phi(\mathbf{v}_2) = \cdots = \phi(\mathbf{v}_6) = 0$, so $\phi(\mathbf{u}) = 0$ for any $\mathbf{u} \in \mathrm{span}(\mathbf{v}_1, \dots, \mathbf{v}_6) = V_6$, but $\phi(\mathbf{v}_7) = -2$, so $\mathbf{v}_7 \not\in V_6$. Thus 7 is a jump.

(8) We have $\mathbf{v}_8 = \mathbf{v}_2 - \mathbf{v}_7 \in V_7$, so 8 is not a jump.

The set of jumps is thus $\{1, 2, 5, 7\}$.