Vector Spaces and Fourier Theory — Problem Sheet 7

Solution 1. Note that $V \cap W$ is the set of matrices of the form $\left[\begin{smallmatrix} a & b \\ b & -a \end{smallmatrix}\right]$, so if we put $u_1 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ and $u_2 = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ then u_1, u_2 is a basis for $V \cap W$. We now put $v_1 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and $w_1 = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$. We find that u_1, u_2, v_1 is a basis for V, and u_1, u_2, w_1 is a basis for W.

Solution 2.

$$\begin{aligned} \dim(U+V) &= \dim(U) + \dim(V) - \dim(U \cap V) = 2 + 3 - 1 = 4 \\ \dim(V+W) &= \dim(V) + \dim(W) - \dim(V \cap W) = 3 + 4 - 2 = 5 \\ \dim(U+V+W) &= \dim(U+V) + \dim(W) - \dim((U+V) \cap W) = 4 + 4 - 3 = 5. \end{aligned}$$

Now it is clear that $V + W \leq U + V + W$ and $\dim(V+W) = \dim(U+V+W)$; this can only happen if $V + W = U + V + W$. It is also clear that $U \leq U + V + W$ but $U + V + W = V + W$ so $U \leq V + W$ as claimed.

Solution 3. (a) Here we have

$$\phi \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = [x, x^2] \left[\begin{smallmatrix} a+bx \\ c+dx \end{smallmatrix}\right] = ax + (b+c)x^2 + dx^3$$

It follows that the list $v_1 = x, v_2 = x^2, v_3 = x^3$ is a basis for image(ϕ), and the matrices $u_1 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right], u_2 = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right], u_3 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ have $\phi(u_i) = v_i$. We also see that $\phi \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = 0$ iff $a = d = 0$ and $c = -b$, so $u_4 = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ gives a basis for $\ker(\phi)$. Finally, we can take $v_4 = 1$ to extend our list $v_1, \dots, v_3$ to a basis for all of $\mathbb{R}[x]_{\leq 3}$. Our final answer is thus:

$$u_1 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right] \qquad u_2 = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] \qquad u_3 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right] \qquad u_4 = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$$

$$v_1 = x \qquad v_2 = x^2 \qquad v_3 = x^3 \qquad v_4 = 1$$

(b) Here we have

$$\psi(ax^2 + bx + c) = \begin{bmatrix} a+b+c \\ a-b+c \\ b \end{bmatrix} = (a+c) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

From this it is clear that the list $v_1 = [1, 1, 0]^T, v_2 = [1, -1, 1]^T$ is a basis for image(ψ), and that if we put $u_1 = 1$ and $u_2 = x$ then $\phi(u_i) = v_i$ for $i = 1, 2$. Moreover, we have $\phi(ax^2 + bx + c = 0$ iff $b = 0$ and $c = -a$, so $u_3 = x^2 - 1$ gives a basis for $\ker(\phi)$. Finally, almost any choice of v_3 will ensure that v_1, v_2, v_3 is a basis of $\mathbb{R}^3$, but the simplest is to take $v_3 = [1, 0, 0]^T$. Our final answer is

$$u_1 = 1 \qquad u_2 = x \qquad u_3 = x^2 - 1$$

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

(c) Here we have

$$\chi \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = \left[\begin{smallmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & -a-d \end{smallmatrix}\right] = a \left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right] + b \left[\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right] + c \left[\begin{smallmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right] + d \left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right]$$

It follows that we can take

$$u_1 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right] u_2 = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] u_3 = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right] u_4 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$$

$$v_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad v_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

and then $\chi(u_i) = v_i$ and $v_1, \dots, v_4$ is a basis for image(χ). Moreover, it is clear that $\chi(A)$ can only be zero if A is zero, so $\ker(\chi) = 0$, so no more u's need to be added. On the other hand, we need five more v's to make up a basis for $M_3\mathbb{R}$. The obvious choices are as follows:

$$v_5 = \left[\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right] \quad v_6 = \left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right] \quad v_7 = \left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right] \quad v_8 = \left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right] \quad v_9 = \left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right]$$

(d) Here we have

$$\theta([a,b,c,d]^T) = ax^2 + b(x+1)^2 + c(x-1)^2 + d(x^2+1) = (a+b+c+d)x^2 + 2(b-c)x + (b+c+d)$$

From this we find that

$$\begin{aligned} \theta([1,0,0,0]^T) &= x^2 \\ \theta([0,1/4,-1/4,0]^T) &= x \\ \theta([-1,0,0,1]^T) &= 1 \\ \theta([0,1,1,-2]^T) &= 0. \end{aligned}$$

We can therefore take

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad u_2 = \begin{bmatrix} 0 \\ 1/4 \\ -1/4 \\ 0 \end{bmatrix} \quad u_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad u_4 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ -2 \end{bmatrix}$$

$$v_1 = x^2 \qquad v_2 = x \qquad v_3 = 1$$

Solution 4. (a) If $a \in \ker(\pi)$ then $\pi(a) = 0$ so $a_0 = a_1 = 0$. Using the relation $a_2 = 3a_1 - 2a_0$ we see that $a_2 = 0$. We can now use the relation $a_3 = 3a_2 - 2a_1$ to see that $a_3 = 0$. More generally, if $a_0 = \cdots = a_{i-1} = 0$ (for some $i \geq 2$) then the relation $a_i = 3a_{i-1} - 2a_{i-2}$ tells us that $a_i = 0$ as well. It follows by induction that $a_i = 0$ for all i, so $a = 0$. Thus $\ker(\pi) = 0$ as claimed.

(b) We have $u_{i+2} - 3u_{i+1} + 2u_i = 1 - 3 + 2 = 0$, so $u \in V$. We also have

$$v_{i+2} - 3v_{i+1} + 2v_i = 2^{i+2} - 3.2^{i+1} + 2.2^i = 2^i(4 - 3.2 + 2) = 0,$$

so $v \in V$.

(c) We have $\pi(u) = [1,1]^T$ and $\pi(v) = [1,2]^T$. By inspection we have $\pi(2u - v) = [1,0]^T$ and $\pi(v - u) = [0,1]^T$, so we can take $b = 2u - v$ and $c = v - u$.

(d) Suppose we have an element $a \in V$. We then have

$$\pi(a - a_0b - a_1c) = \pi(a) - a_0\pi(b) - a_1\pi(c) = [a_0, a_1]^T - a_0[1,0]^T - a_1[0,1]^T = [0,0]^T.$$

As π is injective, this means that $a = a_0b + a_1c$. It is clear that this expression for a in terms of b and c is unique, so b and c give a basis. Next, for a as above we have

$$a = a_0b + a_1c = a_0(2u - v) + a_1(v - u) = (2a_0 - a_1)u + (a_1 - a_0)v,$$

which is a linear combination of u and v. This shows that u and v span V, and they are clearly independent, so they also form a basis.

(a) We have

$$\begin{aligned} \lambda(u) &= \lambda(1,1,1,1,1,\dots) = (1,1,1,1,\dots) = u \\ \lambda(v) &= \lambda(1,2,4,8,16,\dots) = (2,4,8,16,\dots) = 2v \end{aligned}$$

so the matrix of λ with respect to u, v is just $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.