Vector Spaces and Fourier Theory — Problem Sheet 8

Solution 1. (a)

$$\begin{aligned}
\langle C_1, C_1\rangle &= 1^2+1^2+1^2+0^2+1^2+1^2+0^2+0^2+1^2 = 6\\
\langle C_1, C_2\rangle &= 1.1+1.2+1.3+0.2+1.1+1.2+0.3+0.2+1.1 = 10\\
\langle C_1, C_3\rangle &= 1.0+1.1+1.2+0.(-1)+1.0+1.3+0.(-2)+0.(-3)+1.0 = 6\\
\langle C_2, C_2\rangle &= 1^2+2^2+3^2+2^2+1^2+2^2+3^2+2^2+1^2 = 37\\
\langle C_2, C_3\rangle &= 1.0+2.1+3.2+2.(-1)+1.0+2.3+3.(-2)+2.(-3)+1.0 = 0\\
\langle C_3, C_3\rangle &= 0^2+1^2+2^2+(-1)^2+0^2+3^2+(-2)^2+(-3)^2+0^2 = 28
\end{aligned}$$

(b) Suppose that $A^T = A$ and $B^T = -B$. We have $\langle A, B\rangle = \operatorname{trace}(AB^T) = -\operatorname{trace}(AB)$. Using the rules $\operatorname{trace}(X) = \operatorname{trace}(X^T)$ and $\operatorname{trace}(YZ) = \operatorname{trace}(ZY)$ we see that

$$\operatorname{trace}(AB) = \operatorname{trace}((AB)^T) = \operatorname{trace}(B^TA^T) = \operatorname{trace}((-B)A) = -\operatorname{trace}(BA) = -\operatorname{trace}(AB)$$

This means that $\operatorname{trace}(AB) = 0$, so $\langle A, B\rangle = 0$. More directly, we have

$$A = \begin{bmatrix} a_1 & a_2 & a_3\\ a_2 & a_4 & a_5\\ a_3 & a_5 & a_6\end{bmatrix} \qquad B = \begin{bmatrix} 0 & b_1 & b_2\\ -b_1 & 0 & b_3\\ -b_2 & -b_3 & 0\end{bmatrix}$$

for some $a_1, \dots, a_6, b_1, b_2, b_3$. It follows that

$$AB^T = \begin{bmatrix} a_2b_1+a_3b_2 & a_3b_3-a_1b_1 & -a_1b_2-a_2b_3\\ a_4b_1+a_5b_2 & a_5b_3-a_2b_1 & -a_2b_2-a_4b_3\\ a_5b_1+a_6b_2 & a_6b_3-a_3b_1 & -a_3b_2-a_5b_3\end{bmatrix},$$

and the trace of this matrix is zero as required.

(c) V is the set of matrices of the form

$$B = \begin{bmatrix} a & a & a\\ b & b & b\\ c & c & c\end{bmatrix} = a\begin{bmatrix} 1 & 1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} + b\begin{bmatrix} 0 & 0 & 0\\ 1 & 1 & 1\\ 0 & 0 & 0\end{bmatrix} + c\begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 1 & 1 & 1\end{bmatrix}$$

Thus, if we put

$$B_1 = \begin{bmatrix} 1 & 1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \qquad B_2 = \begin{bmatrix} 0 & 0 & 0\\ 1 & 1 & 1\\ 0 & 0 & 0\end{bmatrix} \qquad B_3 = \begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 1 & 1 & 1\end{bmatrix}$$

then $V = \operatorname{span}(B_1, B_2, B_3)$. Now consider a matrix

$$A = \begin{bmatrix} a_1 & a_2 & a_3\\ a_4 & a_5 & a_6\\ a_7 & a_8 & a_9\end{bmatrix},$$

and suppose that $A \in V^\perp$. We then have $0 = \langle A, B_1\rangle = a_1 + a_2 + a_3$ and $0 = \langle A, B_2\rangle = a_4 + a_5 + a_6$ and $0 = \langle A, B_3\rangle = a_7 + a_8 + a_9$. It follows that

$$A\begin{bmatrix} 1\\ 1\\ 1\end{bmatrix} = \begin{bmatrix} a_1+a_2+a_3\\ a_4+a_5+a_6\\ a_7+a_8+a_9\end{bmatrix} = \begin{bmatrix} 0\\ 0\\ 0\end{bmatrix},$$

as claimed.

Solution 2. (a) We have $(x+1)(x^2+x) = x^3+2x^2+x$, so

$$\langle x+1, x^2+x\rangle = \int_{-1}^{1} x^3+2x^2+x\,dx = \left[\tfrac{1}{4}x^4 + \tfrac{2}{3}x^3 + \tfrac{1}{2}x^2\right]_{-1}^{1} = (\tfrac{1}{4}+\tfrac{2}{3}+\tfrac{1}{2}) - (\tfrac{1}{4}-\tfrac{2}{3}+\tfrac{1}{2}) = 4/3.$$

(b) In general, we have

$$\langle x^i, x^j\rangle = \int_{-1}^{1} x^{i+j}\,dx = \left[\frac{x^{i+j+1}}{i+j+1}\right]_{-1}^{1} = \frac{1}{i+j+1} - \frac{(-1)^{i+j+1}}{i+j+1}.$$

If $i+j$ is odd then $i+j+1$ is even and so $(-1)^{i+j+1} = 1$ and $\langle x^i, x^j\rangle = 0$.

(c) Consider a polynomial $f(x) = ax^2+bx+c$. We then have

$$4f(-1) - 8f(0) + 4f(1) = 4(a-b+c) - 8c + 4(a+b+c) = 8a.$$

On the other hand, we have

$$\begin{aligned}\langle f,u\rangle &= \int_{-1}^{1}(ax^2+bx+c)(px^2+q)\,dx \\ &= \int_{-1}^{1} apx^4+bpx^3+(aq+cp)x^2+bqx+cq\,dx \\ &= \left[\tfrac{ap}{5}x^5+\tfrac{bp}{4}x^4+\tfrac{aq+cp}{3}x^3+\tfrac{bq}{2}x^2+cqx\right]_{-1}^{1} \\ &= 2\tfrac{ap}{5}+2\tfrac{aq+cp}{3}+2cq \\ &= (\tfrac{2}{5}p+\tfrac{2}{3}q)a+(\tfrac{2}{3}p+2q)c.\end{aligned}$$

For this to agree with $4f(-1)-8f(0)+4f(1)=8a$, we must have $\frac{2}{5}p+\frac{2}{3}q=8$ and $\frac{2}{3}p+2q=0$. The second of these gives $p=-3q$, which we substitute in the first to get $-\frac{6}{5}q+\frac{2}{3}q=8$ and thus $q=-15$. The equation $p=-3q$ now gives $p=45$, so $u(x)=45x^2-15$.

Solution 3. We use the standard inner product on $C[-1,1]$, given by $\langle f,g\rangle=\int_{-1}^{1}f(x)g(x)\,dx$. Take $g(x)=\sqrt{1-x^2}$, so

$$\|g\|^2=\int_{-1}^{1}g(x)^2\,dx=\int_{-1}^{1}1-x^2\,dx=\left[x-\tfrac{1}{3}x^3\right]_{-1}^{1}=4/3,$$

so $\|g\|=2/\sqrt{3}$. The Cauchy-Schwartz inequality now tells us that $|\langle f,g\rangle|\leq\frac{2}{\sqrt{3}}\|f\|$, or in other words

$$\left|\int_{-1}^{1}\sqrt{1-x^2}\,f(x)\,dx\right|\leq\frac{2}{\sqrt{3}}\left(\int_{-1}^{1}f(x)^2\,dx\right)^{1/2}$$

as claimed. This is an equality iff f is a constant multiple of g. In particular, it is an equality when $f(x)=g(x)=\sqrt{1-x^2}$.

Solution 4. We use the standard inner product on $C[0,1]$, given by $\langle f,g\rangle=\int_0^1 f(x)g(x)\,dx$. The Cauchy-Schwartz inequality says that for any f and g we have $\langle f,g\rangle^2\leq\|f\|^2\|g\|^2=\langle f,f\rangle\langle g,g\rangle$. Now take $g(x)=f(x)^2$, so

$$\begin{aligned}\langle f,g\rangle &= \int_0^1 f(x)^3\,dx \\ \langle f,f\rangle &= \int_0^1 f(x)^2\,dx \\ \langle g,g\rangle &= \int_0^1 f(x)^4\,dx.\end{aligned}$$

The inequality therefore says

$$\left(\int_0^1 f(x)^3\,dx\right)^2\leq\left(\int_0^1 f(x)^2\,dx\right)\left(\int_0^1 f(x)^4\,dx\right)$$

as claimed. This is an equality iff g is a constant multiple of f, so there is a constant c such that $f^2=cf$, so $(f(x)-c)f(x)=0$. If $f(x)$ is nonzero for all x we can divide by $f(x)$ to see that $f(x)=c$ for all x, so f is constant. The same holds by a slightly more complicated argument even if we do not assume that f is everywhere nonzero.

Solution 5. If $A = \left[\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} b_1 & b_2 \\ b_3 & b_4 \end{smallmatrix}\right]$ then

$$\begin{aligned}
\det(A-B) &= \det\left[\begin{smallmatrix} a_1-b_1 & a_2-b_2 \\ a_3-b_3 & a_4-b_4 \end{smallmatrix}\right] = (a_1-b_1)(a_4-b_4) - (a_2-b_2)(a_3-b_3) \\
&= a_1a_4 - a_1b_4 - a_4b_1 + b_1b_4 - a_2a_3 + a_2b_3 + a_3b_2 - b_2b_3 \\
\det(A+B) &= \det\left[\begin{smallmatrix} a_1+b_1 & a_2+b_2 \\ a_3+b_3 & a_4+b_4 \end{smallmatrix}\right] = (a_1+b_1)(a_4+b_4) - (a_2+b_2)(a_3+b_3) \\
&= a_1a_4 + a_1b_4 + a_4b_1 + b_1b_4 - a_2a_3 - a_2b_3 - a_3b_2 - b_2b_3 \\
2\,\mathrm{trace}(A)\,\mathrm{trace}(B) &= 2(a_1+a_4)(b_1+b_4) = 2a_1b_1 + 2a_1b_4 + 2a_4b_1 + 2a_4b_4 \\
\langle A, B\rangle &= -2a_1b_4 - 2a_4b_1 + 2a_2b_3 + 2a_3b_2 + 2a_1b_1 + 2a_1b_4 + 2a_4b_1 + 2a_4b_4 \\
&= 2(a_1b_1 + a_2b_3 + a_3b_2 + a_4b_4).
\end{aligned}$$

(a) We now see that

$$\begin{aligned}
\langle A+B, C\rangle &= 2((a_1+b_1)c_1 + (a_2+b_2)c_3 + (a_3+b_3)c_2 + (a_4+b_4)c_4) \\
&= 2(a_1c_1 + a_2c_3 + a_3c_2 + a_4c_4) + 2(b_1c_1 + b_2c_3 + b_3c_2 + b_4c_4) \\
&= \langle A, C\rangle + \langle B, C\rangle
\end{aligned}$$

(b) Similarly

$$\begin{aligned}
\langle tA, B\rangle &= 2(ta_1b_1 + ta_2b_3 + ta_3b_2 + ta_4b_4) \\
&= t.2(a_1b_1 + a_2b_3 + a_3b_2 + a_4b_4) = t\langle A, B\rangle
\end{aligned}$$

(c) It is clear from the formula $\langle A, B\rangle = 2(a_1b_1 + a_2b_3 + a_3b_2 + a_4b_4)$ that $\langle A, B\rangle = \langle B, A\rangle$.
(d) In general we have $\langle A, A\rangle = 2a_1^2 + 4a_2a_3 + 2a_4^2$. If we take $A = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ (so $a_1 = a_4 = 0$ and $a_2 = 1$ and $a_3 = -1$) then $\langle A, A\rangle = -4 < 0$.
(e) However, if $A \in V$ then $a_3 = a_2$ so $\langle A, A\rangle = 2a_1^2 + 4a_2^2 + 2a_4^2$. This is always nonnegative, and can only be zero if $a_1 = a_2 = a_4 = 0$, which means that $A = 0$ (because $a_3 = a_2$).

Solution 6. (a) We have

$$\langle f, g\rangle' = (fg + f'g')' = f'g + fg' + f''g' + f'g'' = (g + g'')f' + (f + f'')g',$$

and this is zero because $f + f'' = 0 = g + g''$. Thus $\langle f, g\rangle' = 0$.
(b) If $f \in V$ then $f + f'' = 0$. Differentiating this gives $f' + f''' = 0$, which shows that $f' \in V$.
(c) It is clear that $\langle f, g\rangle = \langle g, f\rangle$ and $\langle f + g, h\rangle = \langle f, h\rangle + \langle g, h\rangle$ and $\langle tf, g\rangle = t\langle f, g\rangle$. All that is left is to show that $\langle f, f\rangle \geq 0$, with equality only when $f = 0$. For this, we note that

$$\begin{aligned}
\langle \sin, \sin\rangle &= \sin^2 + \cos^2 = 1 \\
\langle \cos, \cos\rangle &= \cos^2 + (-\sin)^2 = 1 \\
\langle \sin, \cos\rangle &= \sin\cos + \cos.(-\sin) = 0.
\end{aligned}$$

Any element $f \in V$ can be written as $f = a.\sin + b.\cos$ for some $a, b \in \mathbb{R}$, and we deduce that

$$\langle f, f\rangle = a^2\langle \sin, \sin\rangle + 2ab\langle \sin, \cos\rangle + b^2\langle \cos, \cos\rangle = a^2 + b^2.$$

From this it is clear that $\langle f, f\rangle \geq 0$, with equality iff $a = b = 0$, or equivalently $f = 0$.
(d) We have

$$\begin{aligned}
D(\sin) &= \cos & &= 0.\sin + 1.\cos \\
D(\cos) &= -\sin & &= -1.\sin + 0.\cos
\end{aligned}$$

It follows that the matrix of D is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Solution 7. The obvious basis for V consists of the matrices

$$P_1 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right] \qquad P_2 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right] \qquad P_3 = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$$

Using the fact that

$$\langle \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right], \left[\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right]\rangle = ap + bq + cr + ds,$$

we see that the sequence P_1, P_2, P_3 is orthogonal, with $\langle P_1, P_1\rangle = \langle P_2, P_2\rangle = 1$ and $\langle P_3, P_3\rangle = 2$. This means that

$$\pi(A) = \langle A, P_1\rangle P_1 + \langle A, P_2\rangle P_2 + \tfrac{1}{2}\langle A, P_3\rangle P_3.$$

Now take $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. We find that $\langle A, P_1\rangle = a$ and $\langle A, P_2\rangle = d$ and $\langle A, P_3\rangle = b + c$, so we get

$$\pi(A) = aP_1 + dP_2 + \tfrac{1}{2}(b + c)P_3 = \left[\begin{smallmatrix} a & (b+c)/2 \\ (b+c)/2 & d \end{smallmatrix}\right].$$

On the other hand, we have

$$A + A^T = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] + \left[\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right] = \left[\begin{smallmatrix} 2a & b+c \\ b+c & 2d \end{smallmatrix}\right],$$

so $\pi(A) = (A + A^T)/2$.