Vector Spaces and Fourier Theory — Problem Sheet 9

Solution 1. Put $W = \text{span}(\mathcal{W})$ and

$$\pi(v) = \sum_i \frac{\langle v, w_i\rangle}{\langle w_i, w_i\rangle} w_i$$

and $\epsilon(v) = v - \pi(v)$. We saw in lectures that $v = \pi(v) + \epsilon(v)$, with $\pi(v) \in W$ and $\epsilon(v) \in W^\perp$, so $\|v\|^2 = \|\pi(v)\|^2 + \|\epsilon(v)\|^2 \geq \|\pi(v)\|^2$. On the other hand, the vectors $\langle v, w_i\rangle w_i / \langle w_i, w_i\rangle$ are orthogonal to each other, so Pythagoras tells us that

$$\|v\|^2 = \sum_i \left\| \frac{\langle v, w_i\rangle}{\langle w_i, w_i\rangle} w_i \right|^2 = \sum_i \frac{\langle v, w_i\rangle^2}{\langle w_i, w_i\rangle^2} \|w_i\|^2 = \sum_i \frac{\langle v, w_i\rangle^2}{\langle w_i, w_i\rangle^2} \langle w_i, w_i\rangle = \sum_i \frac{\langle v, w_i\rangle^2}{\langle w_i, w_i\rangle}.$$

Alternatively, we can introduce the orthonormal sequence $\widehat{w}_i = w_i / \|w_i\|$. Parseval's inequality tells us that

$$\|v\|^2 \geq \sum_i \langle v, \widehat{w}_i\rangle^2 = \sum_i \left\langle v, \frac{w_i}{\|w_i\|} \right\rangle^2 = \sum_i \frac{\langle v, w_i\rangle^2}{\|w_i\|^2} = \sum_i \frac{\langle v, w_i\rangle^2}{\langle w_i, w_i\rangle}$$

Solution 2. Observe that the integrals involved in the claimed inequality can be interpreted as follows:

$$\int_{-1}^{1} f(x)^2\, dx = \langle f, f\rangle$$
$$\int_{-1}^{1} f(x)\, dx = \langle f, 1\rangle$$
$$\int_{-1}^{1} x f(x)\, dx = \langle f, x\rangle.$$

Note also that $\langle 1, x\rangle = \int_{-1}^{1} x\, dx = 0$, so the sequence $\mathcal{W} = 1, x$ is strictly orthogonal. We can therefore apply Exercise **??**: it tells us that

$$\langle f, f\rangle \geq \frac{\langle f, 1\rangle^2}{\langle 1, 1\rangle} + \frac{\langle f, x\rangle^2}{\langle x, x\rangle}.$$

Here $\langle 1, 1\rangle = \int_{-1}^{1} 1\, dx = 2$ and $\langle x, x\rangle = \int_{-1}^{1} x^2\, dx = 2/3$, so we get

$$\int_{-1}^{1} f(x)^2\, dx \geq \tfrac{1}{2}\left(\int_{-1}^{1} f(x)\, dx\right)^2 + \tfrac{3}{2}\left(\int_{-1}^{1} x\, f(x)\, dx\right)^2.$$

We can now multiply by two to get the inequality in the question.

Solution 3. Consider the matrices

$$B_1 = A_1 = \begin{bmatrix} 0&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&0 \end{bmatrix} \qquad B_2 = A_2 - A_1 = \begin{bmatrix} 0&0&0&0\\ 0&0&1&0\\ 0&1&0&0\\ 0&0&0&0 \end{bmatrix} \qquad B_3 = A_3 - A_2 = \begin{bmatrix} 0&1&1&0\\ 1&0&0&1\\ 1&0&0&1\\ 0&1&1&0 \end{bmatrix} \qquad B_4 = A_4 - A_3 = \begin{bmatrix} 1&0&0&1\\ 0&0&0&0\\ 0&0&0&0\\ 1&0&0&1 \end{bmatrix}$$

If $i \neq j$ we see that B_i and B_j do not overlap, or in other words, in any place where B_i has a one, B_j has a zero. To calculate $\langle B_i, B_j\rangle$ we multiply all the entries in B_i by the corresponding entries in B_j, and add these terms together. All the terms are zero because the matrices do not overlap, so $\langle B_i, B_j\rangle = 0$, so we have an orthogonal sequence. From the formulae

$$B_1 = A_1 \qquad B_2 = A_2 - A_1 \qquad B_3 = A_3 - A_2 \qquad B_4 = A_4 - A_3$$

we deduce that

$$A_1 = B_1 \qquad A_2 = B_1 + B_2 \qquad A_3 = B_1 + B_2 + B_3 \qquad A_4 = B_1 + B_2 + B_3 + B_4.$$

From these two sets of equations together we see that $\text{span}\{B_1, \ldots, B_i\} = \text{span}\{A_1, \ldots, A_i\}$ for all i. We were asked for an orthonormal sequence, so we now put $C_i = B_i / \|B_i\|$. Note that if a matrix X contains only zeros and ones then $\|X\|^2$ is just the number of ones. Using this we see that

$$\|B_1\| = \sqrt{2} \qquad \|B_2\| = \sqrt{2} \qquad \|B_3\| = \sqrt{8} \qquad \|B_4\| = 2,$$

so

$$C_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0 \end{bmatrix} \qquad C_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&0 \end{bmatrix} \qquad C_3 = \frac{1}{\sqrt{8}}\begin{bmatrix} 0&1&1&0\\1&0&0&1\\1&0&0&1\\0&1&1&0 \end{bmatrix} \qquad C_4 = \frac{1}{2}\begin{bmatrix} 1&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&1 \end{bmatrix}$$

Solution 4. The answer is

$$\hat{v}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix} \qquad \hat{v}_2 = \frac{1}{\sqrt{20}}\begin{bmatrix}1\\1\\1\\1\\-4\end{bmatrix} \qquad \hat{v}_3 = \frac{1}{\sqrt{12}}\begin{bmatrix}1\\1\\1\\-3\\0\end{bmatrix} \qquad \hat{v}_4 = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\1\\-2\\0\\0\end{bmatrix} \qquad \hat{v}_5 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\\0\\0\end{bmatrix}$$

The steps are as follows. We will first find a strictly orthogonal sequence $v_1, \dots, v_5$ and then put $\hat{v}_i = v_i/\|v_i\|$. We start with $v_1 = u_1$, which gives $\langle v_1, v_1\rangle = 5$ and $\langle u_2, v_1\rangle = 4$. We then have

$$v_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\langle v_1, v_1\rangle} v_1 = \begin{bmatrix}1\\1\\1\\1\\0\end{bmatrix} - \frac{4}{5}\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix} = \frac{1}{5}\begin{bmatrix}1\\1\\1\\1\\-4\end{bmatrix}.$$

This gives

$$\langle v_2, v_2\rangle = (1^2 + 1^2 + 1^2 + 1^2 + (-4)^2)/25 = 20/25 = 4/5,$$

and $\langle u_3, v_1\rangle = 3$ and $\langle u_3, v_2\rangle = 3/5$ so

$$v_3 = u_3 - \frac{\langle u_3, v_1\rangle}{\langle v_1, v_1\rangle} v_1 - \frac{\langle u_3, v_2\rangle}{\langle v_2, v_2\rangle} v_2 = \begin{bmatrix}1\\1\\1\\0\\0\end{bmatrix} - \frac{3}{5}\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix} - \frac{3/5}{4/5}\frac{1}{5}\begin{bmatrix}1\\1\\1\\1\\-4\end{bmatrix} = \frac{1}{20}\begin{bmatrix}5\\5\\5\\-15\\0\end{bmatrix} = \frac{1}{4}\begin{bmatrix}1\\1\\1\\-3\\0\end{bmatrix}.$$

This gives $\langle v_3, v_3\rangle = (1^2 + 1^2 + 1^2 + (-3)^2)/16 = 3/4$ and $\langle u_4, v_1\rangle = 2$ and $\langle u_4, v_2\rangle = 2/5$ and $\langle u_4, v_3\rangle = 2/4 = 1/2$, so

$$\begin{aligned} v_4 &= u_4 - \frac{\langle u_4, v_1\rangle}{\langle v_1, v_1\rangle} v_1 - \frac{\langle u_4, v_2\rangle}{\langle v_2, v_2\rangle} v_2 - \frac{\langle u_4, v_3\rangle}{\langle v_3, v_3\rangle} v_3 \\ &= \begin{bmatrix}1\\1\\0\\0\\0\end{bmatrix} - \frac{2}{5}\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix} - \frac{2/5}{4/5}\frac{1}{5}\begin{bmatrix}1\\1\\1\\1\\-4\end{bmatrix} - \frac{2/4}{3/4}\frac{1}{4}\begin{bmatrix}1\\1\\1\\-3\\0\end{bmatrix} = \frac{1}{30}\begin{bmatrix}10\\10\\-20\\0\\0\end{bmatrix} = \frac{1}{3}\begin{bmatrix}1\\1\\-2\\0\\0\end{bmatrix}. \end{aligned}$$

This in turn gives $\langle v_4, v_4\rangle = 2/3$ and $\langle u_5, v_1\rangle = 1$ and $\langle u_5, v_2\rangle = 1/5$ and $\langle u_5, v_3\rangle = 1/4$ and $\langle u_5, v_4\rangle = 1/3$ so

$$\begin{aligned} v_5 &= u_5 - \frac{\langle u_5, v_1\rangle}{\langle v_1, v_1\rangle} v_1 - \frac{\langle u_5, v_2\rangle}{\langle v_2, v_2\rangle} v_2 - \frac{\langle u_5, v_3\rangle}{\langle v_3, v_3\rangle} v_3 - \frac{\langle u_5, v_4\rangle}{\langle v_4, v_4\rangle} v_4 \\ &= \begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix} - \frac{1}{5}\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix} - \frac{1/5}{4/5}\frac{1}{5}\begin{bmatrix}1\\1\\1\\1\\-4\end{bmatrix} - \frac{1/4}{3/4}\frac{1}{4}\begin{bmatrix}1\\1\\1\\-3\\0\end{bmatrix} - \frac{1/3}{2/3}\frac{1}{3}\begin{bmatrix}1\\1\\-2\\0\\0\end{bmatrix} = \frac{1}{60}\begin{bmatrix}30\\-30\\0\\0\\0\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\-1\\0\\0\\0\end{bmatrix}. \end{aligned}$$

In summary, the vectors

$$v_1 = \begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix} \qquad v_2 = \frac{1}{5}\begin{bmatrix}1\\1\\1\\1\\-4\end{bmatrix} \qquad v_3 = \frac{1}{4}\begin{bmatrix}1\\1\\1\\-3\\0\end{bmatrix} \qquad v_4 = \frac{1}{3}\begin{bmatrix}1\\1\\-2\\0\\0\end{bmatrix} \qquad v_5 = \frac{1}{2}\begin{bmatrix}1\\-1\\0\\0\\0\end{bmatrix},$$

form an orthogonal (but not yet orthonormal) sequence with $\mathrm{span}\{v_1, \dots, v_i\} = \mathrm{span}\{u_1, \dots, u_i\}$ for $i = 1, \dots, 5$. To get an orthonormal sequence we put $\hat{v}_i = v_i/\|v_i\|$. We have

$$\|v_1\| = \sqrt{5} \qquad \|v_2\| = \sqrt{4/5} \qquad \|v_3\| = \sqrt{3/4} \qquad \|v_4\| = \sqrt{2/3} \qquad \|v_5\| = \sqrt{1/2}.$$

and so

$$\hat{v}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix} \qquad \hat{v}_2 = \frac{1}{\sqrt{20}}\begin{bmatrix}1\\1\\1\\1\\-4\end{bmatrix} \qquad \hat{v}_3 = \frac{1}{\sqrt{12}}\begin{bmatrix}1\\1\\1\\-3\\0\end{bmatrix} \qquad \hat{v}_4 = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\1\\-2\\0\\0\end{bmatrix} \qquad \hat{v}_5 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\\0\\0\end{bmatrix}$$

Solution 5. The resulting orthonormal basis is $\{1, \sqrt{2}t, \sqrt{2}(t^2 - 1/2)\}$. The calculation is as follows. We first note that if $i + j$ is an odd number, then $t^{i+j}e^{-t^2}$ is an odd function, so its integral from $-\infty$ to ∞ is zero, so $\langle t^i, t^j\rangle = 0$. In particular, t is orthogonal to 1 and t^2. Next, the hint tells us that

$$\begin{aligned}\langle 1,1\rangle &= 1\\ \langle t,t\rangle &= 1/2\\ \langle 1,t^2\rangle &= 1/2\\ \langle t^2,t^2\rangle &= 3/4.\end{aligned}$$

It follows that $\|t\| = 1/\sqrt{2}$, so $1, \sqrt{2}t$ is an orthonormal sequence. The projection of t^2 orthogonal to these is

$$t^2 - \langle t^2, 1\rangle 1 - \langle t^2, \sqrt{2}t\rangle\sqrt{2}t = t^2 - 1/2.$$

To normalise this, we note that

$$\langle t^2-1/2, t^2-1/2\rangle = \langle t^2,t^2\rangle - 2\langle t^2, 1/2\rangle + \langle 1/2,1/2\rangle = 3/4 - \langle t^2,1\rangle + \langle 1,1\rangle/4 = \tfrac{3}{4} - \tfrac{1}{2} + \tfrac{1}{4} = 1/2.$$

It follows that $\|\sqrt{2}(t-1/2)\| = 1$, so our orthonormal basis is $\{1, \sqrt{2}t, \sqrt{2}(t^2-1/2)\}$.

Solution 6. (a) Take

$$v_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\\0\end{bmatrix} \qquad v_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\0\\1\\-1\end{bmatrix} \qquad v_3 = \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

so $(x_1-x_2)^2/2 = \langle x, v_1\rangle^2$ and $(x_3-x_4)^2/2 = \langle x, v_2\rangle^2$ and $(x_1+x_2+x_3+x_4)^2/4 = \langle x, v_3\rangle^2$, so

$$\alpha(x) = \|x\|^2 - \sum_{i=1}^{3}\langle x, v_i\rangle^2.$$

It is easy to check that $\langle v_1,v_2\rangle = \langle v_1,v_3\rangle = \langle v_2,v_3\rangle = 0$ and $\langle v_1,v_1\rangle = \langle v_2,v_2\rangle = \langle v_3,v_3\rangle = 1$, so the sequence is orthonormal. Parseval's inequality is thus applicable, and it tells us that $\sum_{i=1}^{3}\langle x,v_i\rangle^2 \leq \|x\|^2$, or in other words $\alpha(x) \geq 0$.

(b) The vector $v_4 = [1,1,-1,-1]^T/2$ is a unit vector orthogonal to v_1, v_2 and v_3, so the sequence v_1, v_2, v_3, v_4 is orthonormal. (How did we find this? If $v_4 = [a,b,c,d]^T$ we must have $a = b$ for orthogonality with v_1, and $c = d$ for orthogonality with v_2, and $a+b+c+d = 0$ for orthogonality with v_3, and $a^2+b^2+c^2+d^2 = 1$ to make v_4 a unit vector. The only two solutions are $[1,1,-1,-1]^T/2$ and $[-1,-1,1,1]^T/2$, and either of these will do.)

(c) By direct calculation, we have

$$\begin{aligned}\alpha(x) =& x_1^2+x_2^2+x_3^2+x_4^2 - \tfrac{1}{4}x_1^2 - \tfrac{1}{4}x_2^2 - \tfrac{1}{4}x_3^2 - \tfrac{1}{4}x_4^2\\ &- \tfrac{1}{2}x_1x_2 - \tfrac{1}{2}x_1x_3 - \tfrac{1}{2}x_1x_4 - \tfrac{1}{2}x_2x_3 - \tfrac{1}{2}x_2x_4 - \tfrac{1}{2}x_3x_4\\ &- \tfrac{1}{2}x_1^2 - \tfrac{1}{2}x_2^2 + x_1x_2 - \tfrac{1}{2}x_3^2 - \tfrac{1}{2}x_4^2 + x_3x_4\\ =& \tfrac{1}{4}x_1^2 + \tfrac{1}{4}x_2^2 + \tfrac{1}{4}x_3^2 + \tfrac{1}{4}x_4^2 + \tfrac{1}{2}x_1x_2 - \tfrac{1}{2}x_1x_3 - \tfrac{1}{2}x_1x_4 - \tfrac{1}{2}x_2x_3 - \tfrac{1}{2}x_2x_4 + \tfrac{1}{2}x_3x_4\\ =& (x_1+x_2-x_3-x_4)^2/4 = \langle x, v_4\rangle^2.\end{aligned}$$

We could have done this without calculation as follows. The sequence $v_1, \ldots, v_4$ is orthonormal (hence linearly independent) of length 4, so it is a basis for $\mathbb{R}^4$, so x automatically lies in the span. Thus Parseval's inequality for this extended sequence is actually an equality, which means that

$$\|x\|^2 = \langle x,v_1\rangle^2 + \langle x,v_2\rangle^2 + \langle x,v_3\rangle^2 + \langle x,v_4\rangle^2.$$

Rearranging this gives $\alpha(x) = \langle x, v_4\rangle^2$ as before.