Vector Spaces and Fourier Theory — Problem Sheet 10

Solution 1. If $\mathbf{v} = [x, y, z]^T$ we have

$$\langle \phi(\mathbf{v}), A\rangle = \langle \left[\begin{smallmatrix} x & y \\ y & z \end{smallmatrix}\right], \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]\rangle = ax + by + cy + dz = ax + (b+c)y + dz = \langle \left[\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right], \left[\begin{smallmatrix} a \\ b+c \\ d \end{smallmatrix}\right]\rangle,$$

so

$$\phi^*(A) = \mathbf{w} = \left[\begin{smallmatrix} a \\ b+c \\ d \end{smallmatrix}\right].$$

Solution 2. We have

$$\langle \left[\begin{smallmatrix} 0 & a_4 & a_7 \\ 0 & 0 & a_8 \\ 0 & 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{smallmatrix}\right]\rangle = a_4b_2 + a_7b_3 + a_8b_6 = \langle \left[\begin{smallmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 0 & 0 \\ b_2 & 0 & 0 \\ b_3 & b_6 & 0 \end{smallmatrix}\right]\rangle.$$

In other words, if we define

$$\psi \left[\begin{smallmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & 0 & 0 \\ b_2 & 0 & 0 \\ b_3 & b_6 & 0 \end{smallmatrix}\right]$$

we have $\langle \phi(A), B\rangle = \langle A, \psi(B)\rangle$. Thus $\phi^* = \psi$.

Solution 3. We must show that $\phi^* = \phi$, or equivalently that $\langle \phi(A), B\rangle = \langle A, \phi(B)\rangle$ for all $A, B \in M_2\mathbb{R}$. If $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right]$ then

$$\begin{aligned}
\phi(A) &= \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} a+b & a+b \\ c+d & c+d \end{smallmatrix}\right] = (a+b+c+d)\left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right] = (a+b+c+d)Q \\
\langle Q, B\rangle &= \langle \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right]\rangle = p+q+r+s \\
\langle \phi(A), B\rangle &= (a+b+c+d)\langle Q, B\rangle = (a+b+c+d)(p+q+r+s) \\
\phi(B) &= (p+q+r+s)Q \\
\langle A, \phi(B)\rangle &= (p+q+r+s)\langle A, Q\rangle = (p+q+r+s)(a+b+c+d) = \langle \phi(A), B\rangle.
\end{aligned}$$

Solution 4. We have

$$\begin{aligned}
\langle \phi(A), B\rangle &= \operatorname{trace}(\phi(A)B^T) = \operatorname{trace}(AB^T - \tfrac{1}{n}\operatorname{trace}(A)B^T) = \operatorname{trace}(AB^T) - \tfrac{1}{n}\operatorname{trace}(A)\operatorname{trace}(B^T) \\
&= \langle A, B\rangle - \tfrac{1}{n}\operatorname{trace}(A)\operatorname{trace}(B) \\
\langle A, \phi(B)\rangle &= \operatorname{trace}(A\phi(B)^T) = \operatorname{trace}(AB^T - \tfrac{1}{n}\operatorname{trace}(B)A) = \operatorname{trace}(AB^T) - \tfrac{1}{n}\operatorname{trace}(A)\operatorname{trace}(B),
\end{aligned}$$

so $\langle \phi(A), B\rangle = \langle A, \phi(B)\rangle$. (For a slightly more efficient approach, we can note that our expression for $\langle \phi(A), B\rangle$ is symmetric: it does not change if we swap A and B. Thus $\langle \phi(A), B\rangle = \langle \phi(B), A\rangle$, and this is the same as $\langle A, \phi(B)\rangle$ by the axiom $\langle X, Y\rangle = \langle Y, X\rangle$.)

Solution 5. First, if $f(x) = ax^2 + bx + c$ then $f'(x) = 2ax + b$ and $f''(x) = 2a$ for all x, so in particular $\chi(f) = f''(0) = 2a$. This means that $\chi(1) = 0$ and $\chi(x) = 0$ and $\chi(x^2) = 2$.

Next, the element u must have the form $px^2 + qx + r$ for some constants $p, q, r \in \mathbb{R}$. It must satisfy

$$\begin{aligned}
\langle 1, u\rangle &= \chi(1) = 0 \\
\langle x, u\rangle &= \chi(x) = 0 \\
\langle x^2, u\rangle &= \chi(x^2) = 2.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\langle 1, u\rangle &= \int_{-1/2}^{1/2} px^2 + qx + r\,dx = \left[px^3/3 + qx^2/2 + rx\right]_{-1/2}^{1/2} = p/12 + r \\
\langle x, u\rangle &= \int_{-1/2}^{1/2} px^3 + qx^2 + rx\,dx = \left[px^4/4 + qx^3/3 + rx^2/2\right]_{-1/2}^{1/2} = q/12 \\
\langle x^2, u\rangle &= \int_{-1/2}^{1/2} px^4 + qx^3 + rx^2\,dx = \left[px^5/5 + qx^4/4 + rx^3/3\right]_{-1/2}^{1/2} = p/80 + r/12
\end{aligned}$$

so we must have $p/12 + r = 0$ and $q/12 = 0$ and $p/80 + r/12 = 2$. These give $p = 360$ and $q = 0$ and $r = -30$, so $u = 360x^2 - 30$.

Now define $\psi\colon \mathbb{R} \to \mathbb{R}[x]_{\leq 2}$ by $\psi(t) = tu = 360tx^2 - 30t$. We claim that ψ is adjoint to ϕ. Indeed, the standard inner product on $\mathbb{R}$ is just $\langle s, t\rangle = st$, so

$$\langle \phi(f), t\rangle = t\phi(f) = t\langle f, u\rangle = \langle f, tu\rangle = \langle f, \psi(t)\rangle,$$

as required.

Solution 6. (a) First note that $e_n(t) = e^{int}$, so $e_n'(t) = ine^{int}$, so $e_n''(t) = (in)^2 e^{int} = -n^2 e_n(t)$. This means that $\Delta(e_n) = -n^2 e_n$, and thus that $\Delta(f) = \sum_n a_n.(-n^2 e_n) = -\sum_n n^2 a_n e_n$.

(b) Consider elements $f, g \in T_2$, say $f = \sum_{n=-2}^{2} a_n e_n$ and $g = \sum_{k=-2}^{2} b_m e_m$. We then have

$$\begin{aligned}
\Delta(f) &= -\sum_n n^2 a_n e_n \\
\langle \Delta(f), g\rangle &= \left\langle -\sum_n n^2 a_n e_n, \sum_m b_m e_m \right\rangle = -\sum_{n,m} n^2 a_n \, \overline{b_m} \, \langle e_n, e_m\rangle \\
&= -\sum_{n=-2}^{2} n^2 a_n \overline{b_n} \\
\Delta(g) &= -\sum_m m^2 b_m e_m \\
\langle f, \Delta(g)\rangle &= \left\langle \sum_n a_n e_n, -\sum_m m^2 b_m e_m \right\rangle = -\sum_{n,m} m^2 a_n \, \overline{b_m} \, \langle e_n, e_m\rangle \\
&= -\sum_{m=-2}^{2} m^2 a_m \overline{b_m}
\end{aligned}$$

This shows that $\langle \Delta(f), g\rangle = \langle f, \Delta(g)\rangle$, so Δ is self-adjoint.

(c) Part (a) tells us that the matrix of Δ with respect to the standard basis $\mathcal{E} = e_{-2}, e_{-1}, e_0, e_1, e_2$ is

$$D = \begin{bmatrix} -4 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -4 \end{bmatrix}$$

From this it is clear that the eigenvalues are 0, -1 and -4.

(d) The eigenspace for the eigenvalue 0 is spanned by e_0 and so has dimension one. The eigenspace for the eigenvalue -1 is spanned by e_{-1} and e_1, and so has dimension 2. Similarly, the eigenspace for the eigenvalue -4 has dimension two, with basis e_{-2}, e_2.

Solution 7. We can write $f = \sum_{n=-2}^{2} a_n e_n$ for some sequence of coefficients a_n. Note that

$$\begin{aligned}
e_n(0) &= 1 \\
e_n(\pi) &= e^{in\pi} = (-1)^n \\
e_n(-\pi/2) &= e^{-in\pi/2} = (e^{-i\pi/2})^n = (-i)^n \\
e_n(\pi/2) &= e^{in\pi/2} = (e^{i\pi/2})^n = i^n.
\end{aligned}$$

It follows that

$$\begin{aligned}
f(0) &= a_{-2} + a_{-1} + a_0 + a_1 + a_2 \\
f(\pi) &= a_{-2} - a_{-1} + a_0 - a_1 + a_2 \\
f(-\pi/2) &= -a_{-2} + ia_{-1} + a_0 - ia_1 - a_2 \\
f(\pi/2) &= -a_{-2} - ia_{-1} + a_0 + ia_1 - a_2 \\
f(0) - f(\pi) &= 2(a_{-1} + a_1) \\
f(-\pi/2) - f(\pi/2) &= 2i(a_{-1} - a_1).
\end{aligned}$$

As $f(0) = f(\pi)$ and $f(-\pi/2) = f(\pi/2)$, we must have $a_{-1} + a_1 = 0$ and also $a_{-1} - a_1 = 0$, which gives $a_{-1} = a_1 = 0$. We therefore have $f = a_{-2}e_{-2} + a_0 + a_2e_2$, or in other words

$$f(t) = a_{-2}e^{-2it} + a_0 + a_2e^{2it}.$$

This gives

$$f(t + \pi) = a_{-2}e^{-2it}e^{-2\pi i} + a_0 + a_2e^{2it}e^{2\pi i},$$

which is the same as $f(t)$ because $e^{2\pi i} = 1$.

Solution 8. (a) As $f = a_{-2}e_{-2} + a_{-1}e_{-1} + a_0e_0 + a_1e_1 + a_2e_2$ and $e_n(0) = 1$ and $e_n'(0) = in$, we have

$$\begin{aligned} f(0) &= a_{-2} + a_{-1} + a_0 + a_1 + a_2 \\ f'(0) &= i.(-2a_{-2} - a_{-1} + a_1 + 2a_2) \end{aligned}$$

This means that $f \in U$ iff we have

$$\begin{aligned} a_{-2} + a_{-1} + a_0 + a_1 + a_2 &= 0 \\ -2a_{-2} - a_{-1} + a_1 + 2a_2 &= 0 \end{aligned}$$

These equations can be solved in a standard way to give

$$\begin{aligned} a_{-2} &= a_0 + 2a_1 + 3a_2 \\ a_{-1} &= -2a_0 - 3a_1 - 4a_2 \end{aligned}$$

and so

$$\begin{aligned} f &= (a_0 + 2a_1 + 3a_2)e_{-2} - (2a_0 + 3a_1 + 4a_2)e_{-1} + a_0e_0 + a_1e_1 + a_2e_2 \\ &= a_0(e_{-2} - 2e_{-1} + e_0) + a_1(2e_{-2} - 3e_{-1} + e_1) + a_2(3e_{-2} - 4e_{-1} + e_2). \end{aligned}$$

(b) From the last expression above, we observe that the functions

$$\begin{aligned} u_0 &= e_{-2} - 2e_{-1} + e_0 \\ u_1 &= 2e_{-2} - 3e_{-1} + e_1 \\ u_2 &= 3e_{-2} - 4e_{-1} + e_2 \end{aligned}$$

give a basis for U.

(c) Part (b) gives a basis for U of length 3, so $\dim(U) = 3$. We also have a basis $e_{-2}, e_{-1}, e_0, e_1, e_2$ of length 5 for T_2, so $\dim(T_2) = 5$. As $T_2 = U \oplus U^\perp$ we have $\dim(U) + \dim(U^\perp) = \dim(T_2) = 5$, so $\dim(U^\perp) = 5 - 3 = 2$.

(d) Put

$$\begin{aligned} v_0 &= e_{-2} + e_{-1} + e_0 + e_1 + e_2 \\ v_1 &= -2e_{-2} - e_{-1} + e_1 + 2e_2. \end{aligned}$$

As the e_n's are orthonormal, we have

$$\begin{aligned} \langle f, v_0\rangle &= \langle a_{-2}e_{-2} + a_{-1}e_{-1} + a_0e_0 + a_1e_1 + a_2e_2, e_{-2} + e_{-1} + e_0 + e_1 + e_2\rangle \\ &= a_{-2} + a_{-1} + a_0 + a_1 + a_2 = f(0) \\ \langle f, v_1\rangle &= \langle a_{-2}e_{-2} + a_{-1}e_{-1} + a_0e_0 + a_1e_1 + a_2e_2, -2e_{-2} - e_{-1} + e_1 + 2e_2\rangle \\ &= -2a_{-2} - a_{-1} + a_1 + 2a_2 = f'(0). \end{aligned}$$

(e) If $f \in U$ then $\langle f, v_0\rangle = f(0) = 0$ and $\langle f, v_1\rangle = f'(0) = 0$. This shows that v_0 and v_1 lie in $U^\perp$. They are clearly linearly independent, as neither one is a multiple of the other. There are two of them, and $\dim(U^\perp) = 2$ by part (c), so they give a basis for $U^\perp$. Moreover, we have

$$\langle v_0, v_1\rangle = 1.(-1) + 1.(-1) + 1.0 + 1.1 + 1.2 = 0,$$

so they are orthogonal. We can therefore construct an orthonormal basis of $U^\perp$ by dividing v_0 and v_1 by their norms. Explicitly, we have

$$\|v_0\|^2 = 1^2 + 1^2 + 1^2 + 1^2 + 1^2 = 5$$
$$\|v_1\|^2 = (-2)^2 + (-1)^2 + 0^2 + 1^2 + 2^2 = 10$$

so our orthonormal basis consists of the functions

$$\hat{v}_0 = (e_{-2} + e_{-1} + e_0 + e_1 + e_2)/\sqrt{5}$$
$$\hat{v}_1 = (-2e_{-2} - e_{-1} + e_1 + 2e_2)/\sqrt{10}.$$

Solution 9. For any $a, b \in U$ we have

$$\langle \phi(a), \phi(b) \rangle = \langle a, \phi^*\phi(b) \rangle = \langle a, b \rangle.$$

(The first step is the definition of ϕ^*, and the second is the fact that $\phi^*\phi(b) = b$.) In particular, we have $\langle \phi(u_i), \phi(u_j) \rangle = \langle u_i, u_j \rangle$. As the sequence $\mathcal{U}$ is orthonormal, we have $\langle u_i, u_j \rangle = 0$ when $i \neq j$, and $\langle u_i, u_i \rangle = 1$. We therefore see that $\langle \phi(u_i), \phi(u_j) \rangle = 0$ when $i \neq j$, and $\langle \phi(u_i), \phi(u_i) \rangle = 1$, which means that the sequence $\phi(u_1), \ldots, \phi(u_n)$ is also orthonormal.

Solution 10.

(a) We are given that $\phi\phi^*(v) = v$ for all $v \in V$. In particular, we can take $v = \phi(u)$ to get $\phi\phi^*\phi(u) = \phi(u)$, or in other words $\phi(u_1) = \phi(u)$. It follows that $\phi(u_2) = \phi(u - u_1) = \phi(u) - \phi(u_1) = 0$.

(b) We have $\langle \phi^*(a), b \rangle = \langle a, \phi(b) \rangle$ for all $a \in V$ and $b \in U$. In particular we can take $a = \phi(u)$ and $b = u_2$ to get

$$\langle u_1, u_2 \rangle = \langle \phi^*\phi(u), u_2 \rangle = \langle \phi(u), \phi(u_2) \rangle = \langle \phi(u), 0 \rangle = 0.$$

(The penultimate step uses part (a).)

(c) As u_1 and u_2 are orthogonal we have

$$\|u\|^2 = \|u_1 + u_2\|^2 = \|u_1\|^2 + \|u_2\|^2 \geq \|u_1\|^2.$$

(d) We have

$$\|u_1\|^2 = \langle \phi^*\phi(u), \phi^*\phi(u) \rangle = \langle \phi(u), \phi\phi^*\phi(u) \rangle = \langle \phi(u), \phi(u) \rangle = \|\phi(u)\|^2.$$

(Here we have used the equation $\phi\phi^*\phi(u) = \phi(u)$ from part (a).)

(e) If we combine (c) and (d) and take square roots we get $\|\phi(u)\| \leq \|u\|$.