Vector Spaces and Fourier Theory — Problem Sheet 11

Solution 1. (a) We have $\alpha(\mathbf{x}) = A\mathbf{x}$, where $A = \left[\begin{smallmatrix} 1 & 2 & 1 \\ 2 & 9 & 2 \\ 1 & 2 & 1 \end{smallmatrix}\right]$, so A is the matrix that we need.

(b) We have $\alpha = \phi_A$ and so $\alpha^\dagger = \phi_{A^\dagger}$, but clearly $A^\dagger = A$, so α is self-adjoint.

(c) Here we just need to find the eigenvalues and eigenvectors of the matrix A. We have

$$\begin{aligned}
\det(A - tI) &= \det \left[\begin{smallmatrix} 1-t & 2 & 1 \\ 2 & 9-t & 2 \\ 1 & 2 & 1-t \end{smallmatrix}\right] = (1-t)\det\left[\begin{smallmatrix} 9-t & 2 \\ 2 & 1-t \end{smallmatrix}\right] - 2\det\left[\begin{smallmatrix} 2 & 2 \\ 1 & 1-t \end{smallmatrix}\right] + \det\left[\begin{smallmatrix} 2 & 9-t \\ 1 & 2 \end{smallmatrix}\right] \\
&= (1-t)(t^2 - 10t + 5) - 2(-2t) + (t-5) = -t^3 + 11t^2 - 10t \\
&= -t(t^2 - 11t + 10) = -t(t-1)(t-10),
\end{aligned}$$

so the characteristic polynomial is $\det(tI - A) = t(t-1)(t-10)$, so the eigenvalues are 0, 1 and 10. For an eigenvector of eigenvalue 0 we must have

$$\begin{aligned}
x + 2y + z &= 0 \\
2x + 9y + 2z &= 0 \\
x + 2y + z &= 0.
\end{aligned}$$

These equations give $y = 0$ and $z = -x$, so any eigenvector of eigenvalue 0 is a multiple of $[1, 0, -1]^T$. For a unit vector, we can take $u_1 = [1, 0, -1]^T/\sqrt{2}$.

For an eigenvector of eigenvalue 1 we must have

$$\begin{aligned}
x + 2y + z &= x \\
2x + 9y + 2z &= y \\
x + 2y + z &= z.
\end{aligned}$$

These equations give $x = z = -2y$, so any eigenvector of eigenvalue 1 is a multiple of $[-2, 1, -2]^T$. For a unit vector, we can take $u_2 = [-2, 1, -2]^T/3$.

For an eigenvector of eigenvalue 10 we must have

$$\begin{aligned}
x + 2y + z &= 10x \\
2x + 9y + 2z &= 10y \\
x + 2y + z &= 10z.
\end{aligned}$$

These equations give $z = x$ and $y = 4x$, so any eigenvector of eigenvalue 1 is a multiple of $[1, 4, 1]^T$. For a unit vector, we can take $u_3 = [1, 4, 1]^T/(3\sqrt{2})$.

Now u_1, u_2 and u_3 are eigenvectors of a self-adjoint map with different eigenvalues, so they are automatically orthogonal. (It is easy to check this directly, of course.) They are unit vectors, so they give an orthonormal (and so linearly independent) sequence. As this is an independent sequence of length three in a three-dimensional space, it must be a basis.

Solution 2. (a) If $z = [z_0, \ldots, z_4]^T$ and $w = [w_0, \ldots, w_4]^T$ we have

$$\begin{aligned}
\langle \alpha(z), w \rangle &= \langle [z_1, z_2, z_3, z_4, z_0]^T, [w_0, w_1, w_2, w_3, w_4]^T \rangle \\
&= z_1\overline{w_0} + z_2\overline{w_1} + z_3\overline{w_2} + z_4\overline{w_3} + z_0\overline{w_4} \\
&= z_0\overline{w_4} + z_1\overline{w_0} + z_2\overline{w_1} + z_3\overline{w_2} + z_4\overline{w_3} \\
&= \langle [z_0, z_1, z_2, z_3, z_4]^T, [w_4, w_0, w_1, w_2, w_3]^T \rangle.
\end{aligned}$$

Thus, if we define $\beta\colon \mathbb{C}^5 \to \mathbb{C}^5$ by

$$\beta([w_0, w_1, w_2, w_3, w_4]^T) = [w_4, w_0, w_1, w_2, w_3]^T,$$

we have $\langle \alpha(z), w \rangle = \langle z, \beta(w) \rangle$. This means that $\beta = \alpha^\dagger$. We also have

$$\begin{aligned}
\beta\alpha(z) &= \beta([z_1, z_2, z_3, z_4, z_0]^T) = z \\
\alpha\beta(w) &= \alpha([w_4, w_0, w_1, w_2, w_3]^T) = w,
\end{aligned}$$

so β is inverse to α. Thus $\alpha^{-1} = \beta = \alpha^\dagger$.

(b) Suppose that $\alpha(z) = \lambda\, z$, or in other words

$$[z_1, z_2, z_3, z_4, z_0] = [\lambda z_0, \lambda z_1, \lambda z_2, \lambda z_3, \lambda z_4].$$

This means that

$$\begin{aligned} z_1 &= \lambda z_0 \\ z_2 &= \lambda z_1 = \lambda^2 z_0 \\ z_3 &= \lambda z_2 = \lambda^3 z_0 \\ z_4 &= \lambda z_3 = \lambda^4 z_0 \\ z_0 &= \lambda z_4 = \lambda^5 z_0. \end{aligned}$$

The last equation gives $(\lambda^5 - 1)z_0 = 0$. If $\lambda^5 \neq 1$ then we see that $z_0 = 0$, and by substituting this into our other equations we see that $z_1 = z_2 = z_3 = z_4 = 0$ as well, so $z = 0$. Thus, for such λ there are no nonzero eigenvectors, so λ is not an eigenvalue.

Suppose instead that $\lambda^5 = 1$, which means that $\lambda = e^{2\pi ik/5}$ for some $k \in \{0, 1, 2, 3, 4\}$. We then see that the vector

$$u_k = [1, \lambda, \lambda^2, \lambda^3, \lambda^4]^T$$

is an eigenvector of eigenvalue λ. Thus, the eigenvalues are precisely the fifth roots of unity.

Solution 3. (a) If $f = ax^2 + bx + c$ then $f'' = 2a$, so $\alpha(f) = 6ax^2 - 2a$, so $\alpha(f)'' = 12a$, so $\alpha(\alpha(f)) = (3x^2 - 1)\alpha(f)'' = 36ax^2 - 12a = 6\alpha(f)$.

(b) If $\alpha(f) = \lambda f$ then $\alpha(\alpha(f)) = \alpha(\lambda f) = \lambda\alpha(f) = \lambda.\lambda f = \lambda^2 f$. On the other hand, we also know from (a) that $\alpha(\alpha(f)) = 6\alpha(f) = 6\lambda f$, so $\lambda^2 f = 6\lambda f$, so $(\lambda^2 - 6\lambda)f = 0$. As f was assumed to be nonzero, this means that $\lambda^2 - 6\lambda = 0$, or $\lambda(\lambda - 6) = 0$, so $\lambda = 0$ or $\lambda = 6$.

(c) By part (b) the only possible eigenvalues are 0 and 6. Clearly $\alpha(f) = 0$ iff $f'' = 0$ iff $a = 0$, so the eigenvectors of eigenvalue 0 are just the polynomials of the form $bx + c$. In particular, if we put $u_1 = 1$ and $u_2 = x$ then u_1 and u_2 are eigenvectors, and we have $\langle u_1, u_2\rangle = \int_{-1}^{1} x\, dx = 0$, so they are orthogonal. Next take $u_2 = 3x^2 - 1$. By definition we have $\alpha(f) = u_2.f''$ for all f, so in particular $\alpha(u_2) = u_2.u_2'' = 6u_2$, so u_2 is an eigenvector of eigenvalue 6. We are given that α is self-adjoint, so eigenvectors of different eigenvalues are automatically orthogonal, so u_1, u_2, u_3 is an orthogonal sequence.

Solution 4. (a) We have $\langle \phi(f), g\rangle = \int_{-\infty}^{\infty} x\, f(x)\overline{g(x)}\, dx$ and $\langle f, \phi(g)\rangle = \int_{-\infty}^{\infty} f(x)\overline{x\, g(x)}\, dx = \int_{-\infty}^{\infty} \overline{x}\, f(x)\overline{g(x)}\, dx$. Here x is real so $\overline{x} = x$ so $\langle \phi(f), g\rangle = \langle f, \phi(g)\rangle$, which means that ϕ is self-adjoint.

(b) If $\phi(f) = \lambda\, f$ then $x, f(x) = \lambda\, f(x)$ for all $x \in \mathbb{R}$, so $(x - \lambda)\, f(x) = 0$ for all $x \in \mathbb{R}$. If $x \neq \lambda$ then we can divide by $x - \lambda$ to see that $f(x) = 0$. If $\lambda \notin \mathbb{R}$ then this finishes the argument: x can never be equal to λ, so $f(x) = 0$ for all $x \in \mathbb{R}$, so $f = 0$. If λ is real then we need one extra step: we have seen that $f(x) = 0$ for all $x \neq \lambda$, but f is continuous so it cannot jump away from zero at $x = \lambda$, so $f(\lambda) = 0$ as well, so $f = 0$.

Solution 5. We first note that

$$\gamma \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix} - \begin{bmatrix} b & a \\ d & c \end{bmatrix} = \begin{bmatrix} c - b & d - a \\ a - d & b - c \end{bmatrix}.$$

(a) The standard basis for $M_2\mathbb{R}$ consists of the matrices

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

We have

$$\gamma(E_1) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -E_2 + E_3$$
$$\gamma(E_2) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = -E_1 + E_4$$
$$\gamma(E_3) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = E_1 - E_4$$
$$\gamma(E_4) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = E_2 - E_3$$

so the matrix of γ with respect to E_1, E_2, E_3, E_4 is

$$\begin{bmatrix} 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

(b) From our formulae for $\gamma(E_i)$, we see that the matrices $U = E_1 - E_4 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $V = E_2 - E_3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ form a basis for image(γ), and the matrices $I = E_1 + E_4$ and $T = E_2 + E_3$ form a basis for ker(γ). The matrices $E_1, \dots, E_4$ are orthonormal with respect to the usual inner product, so we have

$$\langle U, I\rangle = \langle E_1 - E_4, E_1 + E_4\rangle = 1.1 + (-1).1 = 0$$
$$\langle U, T\rangle = \langle E_1 - E_4, E_2 + E_3\rangle = 0$$
$$\langle V, I\rangle = \langle E_2 - E_3, E_1 + E_4\rangle = 0$$
$$\langle V, T\rangle = \langle E_2 - E_3, E_2 + E_3\rangle = 1.1 + (-1).1 = 0.$$

This shows that $\ker(\gamma) = \text{span}\{I, T\}$ is orthogonal to $\text{image}(\gamma) = \text{span}\{U, V\}$. As the dimensions of these two subspaces add up to the dimension of the whole space, the subspaces must be orthogonal complements of each other.

(c) One checks that $\gamma(U) = -2V$ and $\gamma(V) = -2U$. It follows that $\gamma^2(U) = 4U$, and so $\gamma^4(U) = \gamma^2(4U) = 16U = 4\gamma^2(U)$. Similarly, we have $\gamma^4(V) = 16V = 4\gamma^2(V)$. As $\gamma(I) = \gamma(T) = 0$, we also have $\gamma^4(I) = 0 = 4\gamma^2(I)$ and $\gamma^4(T) = 0 = 4\gamma^2(T)$. Thus gm^4 and $4\gamma^2$ have the same effect on U, V, I and T, which span $M_2\mathbb{R}$, so $\gamma^4 = 4\gamma^2$. Alternatively, we can just square the matrix in part (a) twice to see that

$$\gamma^2 \sim \begin{bmatrix} 2 & 0 & 0 & -2 \\ 0 & 2 & -2 & 0 \\ 0 & -2 & 2 & 0 \\ -2 & 0 & 0 & 2 \end{bmatrix} \qquad \gamma^4 \sim \begin{bmatrix} 8 & 0 & 0 & -8 \\ 0 & 8 & -8 & 0 \\ 0 & -8 & 8 & 0 \\ -8 & 0 & 0 & 8 \end{bmatrix}$$

which again shows that $\gamma^4 = 4\gamma^2$.

(d) As T and I lie in the kernel of γ, they are eigenvectors with eigenvalue 0. As $\gamma(U) = -2V$ and $\gamma(V) = -2U$ we see that $\gamma(U+V) = -2(U+V)$ and $\gamma(U-V) = 2(U-V)$. It follows that $\{I, T, U-V, U+V\}$ is a basis consisting of eigenvectors.